Bifurcations and catastrophes of dynamical systems with centre dimension one

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Abstract

Elementary catastrophes occur in scalar or gradient systems, but the same catastrophes also underlie the more intricate bifurcations of vector fields, providing a more practical means to locate and identify them than standard bifurcation theory. Here we formalise the concept of these *underlying catastrophes*, proving that it identifies contact-equivalent families, and we extend the concept to difference equations (i.e. maps/diffeomorphisms). We deal only with bifurcations of corank one, and centre dimension one (meaning the system has one eigenvalue equal to zero in the case of a vector field, or equal to one in the case of a map). In this case we prove moreover that these underlying catastrophes identify topological bifurcation classes. It is hoped these results point the way to extending the concept of underlying catastrophes to higher coranks and centre dimensions. We illustrate with some simple examples, including a system of biological reaction diffusion equations whose homogenous steady states are shown to undergo butterfly and star catastrophes.

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1 Introduction

The generic local bifurcations of vector fields with one-dimensional centre manifolds are completely classified under topological equivalence, for instance by the so-called A class [2, Chapter 1, Section 3.1]. These classifications, however, work by restricting a vector field to its centre manifold, and do not provide explicit tools to locate bifurcations in multi-dimensional vector fields with multiple parameters; see for example [7, 11]. More practical methods do exist but each have their own limitations. One approach is to find normal form coefficients using methods that normalise a vector field and achieve center manifold reduction concomitantly, see e.g. [11, Section 5.4], but such methods have been explored for only low codimension bifurcations, and numerical implementation can be hindered by the need to find certain critical eigenvectors. Another practical alternative is Lyapunov-Schimdt reduction, as used extensively in [5], but like centre manifold reduction this involves analytic reduction steps, which do not typically result in explicit formulae to compute the location of bifurcation points, at least again not beyond low codimension cases.

In this paper we prove a more constructive result that topologically characterises generic bifurcations only with respect to their *contact* equivalence class. We prove that this characterisation enables the coordinates and parameter values at which any bifurcation occurs to be calculated directly, by finding their *underlying catastrophes*. In other words, we provide readily solvable conditions for locating specific singularities in parameterized families of vector fields, as well as criteria to verify if their parameters are sufficient to fully unfold any of those singularities. These so-called $\mathcal{B}$-$\mathcal{G}$ conditions (from [8]) are novel in that they do not require centre manifold or normal form reduction to be applied, but still provide, in the cases considered, a complete characterisation of the dynamics in terms of the jet coefficients only.

Underlying catastrophes were proposed in [8] as a way of extending Thom's elementary catastrophes (see e.g. [13, 17, 15]) directly to vector fields, and are so called because any bifurcation would have an elementary catastrophe 'underlying' it, making it easy to find using the associated $\mathcal{B}$-$\mathcal{G}$ conditions. Some connection was made in [9] between these $\mathcal{B}$-$\mathcal{G}$ conditions and the Thom-Boardman classification (see [3, 13]), but here we go much further, showing that the underlying catastrophes define full topological families of bifurcations. Our results can be seen as a consolidation of normalisation methods (used for example in [11]) with the contact equivalence approach, resulting in more practical explicit expressions for locating and characterising bifurcations.

We also consider the parallel problem in maps (difference equations), by establishing similar conditions characterising bifurcations of diffeomorphisms with one simple eigenvalue equal to 1 (and all others not lying on the unit circle). We show that the criteria in this case are essentially the same as for vector fields, but applied to their associated 'displacement' functions instead of the maps themselves.

These results substantially simplify the identification of higher codimension bifurcations in any number of dimensions and with any number of parameters, in both continuous and discrete time dynamical systems, intended to help make the theory of bifurcations and singularities more applicable.

A comment about our use of the terms *catastrophe*, *bifurcation*, and *singularity* may be in order. Whereas Thom's catastrophe theory deals with potentials and their gradient vector fields using right equivalence, the idea of *underlying catastrophes* draws upon this to identify singularities of zeroes of more general vector fields, using contact equivalence; indeed one of the of the present paper is to elucidate these ideas. The terms singularity and bifurcation refer here, respectively, to the point where the derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ has less than full rank (is 'singular'), and to the change in topological equivalence class that occurs as parameters vary (usually due to passing through a singularity).

The layout of the paper is as follows. We first formalise, in Section 2, a concept introduced in [8, 9], that any bifurcation of a vector field has an underlying catastrophe, and we extend the concept to maps. We deal only with the situation of bifurcations involving one non-hyperbolic direction (one eigenvalue equal to 0 in a vector field, or one eigenvalue equal to +1 in a map), thus the centre manifold of the bifurcation has dimension one, a property we describe succinctly here as having 'centre dimension one'. (Here we exclude the case of maps with an eigenvalue equal to −1, but we make a brief remark on this at the end).

In Section 3 we briefly recap the notion of $\mathcal{K}$-(or contact) equivalence (see also e.g. [12, 5, 14]), and use it to refine the definition of underlying catastrophes from [8]. Then in Section 4 we summarise the conditions introduced in [8] to find underlying catastrophes, and prove that they are satisfied uniquely by families of functions with those catastrophes. In Sections 5 and 6, we then use these catastrophes to define bifurcations of centre dimension one for vector fields and maps.

Section 7 sets out conditions for families of vector fields and maps to unfold these bifurcations. Families that satisfy those conditions are named here as *minimal stable families*, and we prove, also in Section 7, that they are sufficient for a family to be equivalent to one of the *principal families* (see Section 5.3). Finally, Section 8 gives the proofs of the main theorems from Section 2, with some technical lemmas placed in Appendix A. Finally a few closing remarks are made in Section 9.

2 Statement of main results

The main result of this paper is to prove calculable criteria to classify certain bifurcations in families of dynamical systems with any number of dimensions and parameters. We give these first for vector fields and then for maps.

The first set of criteria is related to families of vector fields and is given in terms of underlying catastrophes of corank 1, a concept that will be formally defined in Section 3. It can be described briefly as being a selection of equivalence classes under $\mathcal{K}$-equivalence, and is shown to provide criteria that are amenable to calculation in Section 4.

Here and throughout the paper, whenever $F(x,\mu)$ is parameterized family of functions, DF will denote the Jacobian derivative of F with respect to the coordinates x only.

Theorem 2.1. *Let $F(x,\mu)$ be a local r-parameter family of vector fields in $\mathbb{R}^n$ at (x_*,μ_*) such that:*

(i) F unfolds an underlying catastrophe of corank 1 and codimension r at $x = x_*$ for $\mu = \mu_*$;

(ii) $DF(x_*,\mu_*)$ has one simple eigenvalue equal to 0, as well as n_s eigenvalues with negative real part and n_u eigenvalues with positive real part (counting algebraic multiplicity) such that $n_s + n_u = n - 1$.

Then F undergoes a codimension r bifurcation of centre dimension 1, i.e., it is topologically equivalent to the (n_s,n_u)-saddle suspension of a one-dimensional principal family of codimension r given by

$$\dot{z} \;=\; \pm z^{r+1} + \mu_{r-1}\, z^{r-1} + \ldots + \mu_0 \; . \tag{1}$$

The proof of Theorem 2.1 is given in Section 8.2, and relies on showing that its hypotheses imply that F is what we name a *minimal stable family* (see Definition 7.1). The result then follows from Theorem 7.1, which guarantees that such families are topologically equivalent to principal families.

The second main result of this paper provides similar criteria for maps, its proof being similar. The definition of minimal stable family for maps is slightly different (see Definition 7.2), but the concept plays the same role in the proof, guaranteeing equivalence to a principal family. The relevant result in the case of maps is Theorem 7.2.

Theorem 2.2. *Let $\Pi(x,\mu)$ be a local r-parameter family of diffeomorphisms in $\mathbb{R}^n$ at (x_*,μ_*) such that:*

- (i) *the family of displacement functions $F(x,\mu) := \Pi(x,\mu) - x$ unfolds an underlying catastrophe of corank 1 and codimension r at $x = x_*$ for $\mu = \mu_*$;*
- (ii) *$D\Pi(x_*,\mu_*)$ has one simple eigenvalue equal to 1, as well as n_s eigenvalues inside the unit circle and n_u eigenvalues outside the unit circle (counting algebraic multiplicity) such that $n_s + n_u = n - 1$.*

Then Π undergoes a codimension r bifurcation of centre dimension 1, i.e., it is topologically equivalent to the (n_s, n_u)-saddle suspension of a one-dimensional principal family of codimension r given by

$$z \;\mapsto\; z \pm z^{r+1} + \mu_{r-1}\, z^{r-1} + \ldots + \mu_0\,. \tag{2}$$

3 Catastrophes in families of functions

The concept of underlying catastrophes was introduced in [8] as a practical method to find the local bifurcations of vector fields. In [9] it was shown via the Thom-Boardman procedure that these identify the stable germs of mappings corresponding to the corank 1 singularities that generate certain bifurcations, but no general theory was developed for how these underlying catastrophes corresponded to stable bifurcations of vector fields. The way to prove this is through $\mathcal{K}$-equivalence and germs, so we briefly summarize both concepts here, and use them to more formally define the notion of an underlying catastrophe of corank 1.

3.1 Germs and their unfoldings

Definition 3.1. *Let $n, p \in \mathbb{N}$, $x_* \in \mathbb{R}^n$, and U, U' be two open neighbourhoods of x_*. Then $f : U \to \mathbb{R}^p$ and $g : U' \to \mathbb{R}^p$ are said to be germ-equivalent at x_* if there is an open neighbourhood $U'' \subset U \cap U'$ of x_* such that $f|_{U''} = g|_{U''}$. Each equivalence class $[f]$ of f under germ-equivalence at x_* is called the germ of f at x_*.*

From now on, we adopt the notation $[f] : (\mathbb{R}^n, x_*) \to (\mathbb{R}^p, y_*)$ to mean that $[f]$ is a germ at $x_* \in \mathbb{R}^n$ of a function $f : \mathbb{R}^n \to \mathbb{R}^p$ such that $f(x_*) = y_*$.

Definition 3.2. *A k-parameter unfolding of a germ $[f] : (\mathbb{R}^n, x_*) \to (\mathbb{R}^p, y_*)$ is a germ $[\tilde{F}] : (\mathbb{R}^{n+k}, (x_*, 0)) \to (\mathbb{R}^{p+k}, (y_*, 0))$ such that:*

- (i) *a representative $\tilde{F}$ of $[\tilde{F}]$ is of the form $\tilde{F}(x,\mu) = (F(x,\mu), \mu)$;*

(ii) $F(x,0) = f(x)$.

Definition 3.3. *Let* $[\tilde{F}]$ *be a* k*-parameter unfolding of* $[f]$ *and* $[h] : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$. *The pullback of* $[\tilde{F}]$ *by* $[h]$ *is the* l*-parameter unfolding* $[h]^*[\tilde{F}]$ *given by* $(x, \eta) \mapsto (F(x, h(\eta)), \eta)$.

3.2 Smooth germs, local families and $\mathcal{K}$-equivalence

A germ or unfolding is said to be smooth if one of its representatives is smooth. The set of smooth germs from $\mathbb{R}^n$ to $\mathbb{R}^p$ at a point $x_* \in \mathbb{R}^n$ will be denoted by $\mathcal{E}_n^p(x_*)$, and this is a vector space over $\mathbb{R}$ when equipped with operations induced from sum of functions and product of a function by a real number.

To define $\mathcal{K}$-equivalence for smooth germs, we are only interested in germs centered at the origin, as this can always be assumed to hold up to a translation of the coordinate system. In that case, we abbreviate $\mathcal{E}_n^p(0) = \mathcal{E}_n^p$.

Definition 3.4. $[\phi] \in \mathcal{E}_n^n$ *is said to be the germ of a local diffeomorphism at* 0 *if there is one element* ϕ *in the class* $[\phi]$ *for which:*

(i) $\phi(0) = 0$;

(ii) $D\phi(0)$ *is invertible.*

The set of germs of local diffeomorphisms at 0 *on* $\mathbb{R}^n$ *is denoted by* L_n. *It is a group under the natural operation induced by composition, which also induces a right group action on the vector space* $\mathcal{E}_n^p$.

To define $\mathcal{K}$-equivalence, let us introduce a symbol $\mathrm{GL}_p(\mathcal{E}_n)$ that denotes the set of $p \times p$ invertible matrices $M(x)$ whose entries are in $\mathcal{E}_n$. One is effectively required to verify only that $M(0)$ is invertible on account of smoothness.

Definition 3.5. *Two smooth* ***germs*** $[f], [g] : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *are said to be* $\mathcal{K}$*-**equivalent** if there are germs* $[\phi] \in L_n$ *and* $[M] \in GL_p(\mathcal{E}_n)$ *such that* $[g] = [M] \cdot [f] \circ [\phi]$.

In the context of diffeomorphisms or vector fields, as is the case of this paper, we are only interested in the case $n = p$, so that will be assumed henceforth.

Definition 3.6. *Let* $U \subset \mathbb{R}^n$, $\Sigma \subset \mathbb{R}^k$ *be open, and* $F : U \times \Sigma \to \mathbb{R}^n$ *be smooth. For any* (x_*, μ_*) *in the domain of* F, *we say that the germ* $[F]$ *of* F *at* (x_*, μ_*) *is a* ***local*** k***-parameter smooth family of functions at*** (x_*, μ_*). *The germ of* $x \mapsto F(x, \mu_*)$ *is called the critical germ of* $[F]$.

As indicated, to properly expand the definition of $\mathcal{K}$-equivalence to smooth families at an arbitrary point, we will need to translate the coordinates to base them around $(0, 0)$.

Definition 3.7. *We say that two local* k*-parameter* ***smooth families*** $[F]$ *and* $[G]$ *at* (x_*, μ_*) *and* (y_*, η_*), *respectively, are* $\mathcal{K}$*-**equivalent** if:*

(i) the translated germs $[f_0] : z \mapsto F(x_* + z, \mu_*)$ *and* $[g_0] : z \mapsto G(y_* + z, \eta_*)$ *in* $\mathcal{E}_n^p$ *are* $\mathcal{K}$*-equivalent;*

(*ii*) *letting* $[\tilde{F}_0]$ *and* $[\tilde{G}_0]$ *denote, respectively, the unfoldings* $(z,\lambda) \mapsto (F(x_* + z, \mu_* + \lambda), \lambda)$ *and* $(z,\lambda) \mapsto (G(y_* + z, \eta_* + \lambda), \lambda)$ *of* $[f_0]$ *and* $[g_0]$*, there are* $([M],[\phi]) \in GL_n(\mathcal{E}_n) \times L_n$, $[h] \in L_k$, *and smooth unfoldings* $[\tilde{\Phi}]$ *of* $[\phi]$ *and* $[\tilde{Q}]$ *of* $[M]$*, such that*

$$[\tilde{G}_0] = [\tilde{Q}] \cdot \Big([h]^*[\tilde{F}_0]\Big) \circ [\tilde{\Phi}]. \tag{3}$$

Remark: there is a slight abuse of notation in (3), as the unfolding $\tilde{Q}$ takes a pair (z,λ) to $(Q(z,\lambda),\lambda) \in \mathrm{GL}_n(\mathcal{E}_n) \times \mathbb{R}^k$, so it is not immediately obvious how to take the product of $[\tilde{Q}]$ with the rest of the expression, which is of the form $(F_0(\Phi(z,\lambda), h(\lambda)),\lambda) \in \mathbb{R}^n \times \mathbb{R}^k$. This product should be read as occurring only in the first entry, the parameter being carried forward, that is, the resulting germ is that of $(Q(z,\lambda) \cdot F_0(\Phi(z,\lambda), h(\lambda)),\lambda)$.

3.3 Corank of a singularity

We say that f has a corank m singularity at x_* if the derivative $Df(x_*)$ has corank m, that is, $m = n - \operatorname{rank} Df(x_*)$. For simplicity and without loss of generality, we will assume that $(x_*, \mu_*) = (0,0)$ in this section, as this can always be achieved by simple translation.

Proposition 3.1. *Let* $[F]$ *be a local* k*-parameter smooth family at* $(0,0)$ *whose critical germ* $[f] : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *is such that* f *has a corank* $m \le n$ *singularity at* $x = 0$*. Then there are* $[f_c] : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ *and a local* k*-parameter smooth family* $[F_c]$ *having* $[f_c]$ *as its critical germ such that:*

(*i*) $Df_c(0) = 0$;

(*ii*) $[f]$ *is* $\mathcal{K}$*-equivalent to the germ of* $f_e : (x_1, x_2) \mapsto (f_c(x_1), x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ *at the origin;*

(*iii*) *the local family* $[F_e] : (x_1, x_2, \mu) \mapsto (F_c(x_1,\mu), x_2)$ *is* $\mathcal{K}$*-equivalent to* $[F]$.

Any pair (f_c, F_c) *as above is called a core of the pair* (f, F).

Proof. The first two statements correspond exactly to [14, Lemma 13.7]. Hence, from (*ii*), there are $([M],[\phi]) \in GL_n(\mathcal{E}_n) \times L_n$ such that $[f_e] = [M] \cdot [f] \circ [\phi]$, and statement number (*iii*) follows directly by defining $F_e(x_1, x_2, \mu) = M(x_1, x_2) \cdot F(\phi(x_1, x_2), \mu)$, with $\tilde{Q}(x_1, x_2, \mu) = (M(x_1, x_2), \mu)$, $h(\mu) = \mu$, and $\tilde{\Phi}(x_1, x_2, \mu) = (\phi(x_1, x_2), \mu)$. □

This can be restated as follows: if $p = n$, every local k-parameter smooth family at $(0,0)$ whose critical germ has a corank $m \le n$ singularity is $\mathcal{K}$-equivalent to a local k-parameter smooth family of the form $(x_1, x_2) \mapsto (F_c(x_1, \mu), x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, where the critical germ of F_c has rank zero.

3.4 Underlying catastrophes of corank 1

We can now define the central concept of this paper: underlying catastrophes of corank 1. We assume that $n = p$, as we are interested in studying families of vector fields or diffeomorphisms.

Definition 3.8. *Let* $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*. The normal form of an* n*-dimensional underlying catastrophe of corank 1 and codimension* r *is* $\mathcal{C}^n_{1^r}(x,\mu) = (\mathcal{U}_{1^r}(x_1,\mu), x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$*, where* $\mu = (\mu_0, \dots, \mu_{r-1})$ *and*

$$\mathcal{U}_{1^r}(z,\mu) = z^{r+1} + \mu_{r-1} z^{r-1} + \mu_{r-2} z^{r-2} + \ldots + \mu_0\ . \tag{4}$$

Definition 3.9. *A local r-parameter smooth family $[F]$ at (x_*, μ_*) is said to unfold an underlying catastrophe of corank 1 and codimension r at $x = x_*$ for $\mu = \mu_*$ if it is $\mathcal{K}$-equivalent to the local r-parameter family $[\mathcal{C}^n_{1^r}]$ at $(0, 0)$.*

Proposition 3.2. *If the local family $[G]$ is $\mathcal{K}$-equivalent to a local family $[F]$ unfolding an underlying catastrophe of corank 1, then it unfolds the same underlying catastrophe of corank 1.*

Proof. It suffices to notice that $\mathcal{K}$-equivalence is an equivalence relation. □

Each catastrophe as defined above has a singular germ, corresponding to $\mu = 0$ in the normal form, which can be considered to be its organising centre.

Definition 3.10. *$[f] \in \mathcal{E}^n_n(x_*)$ is said to be an underlying catastrophe germ of corank 1 and codimension r if its translated germ $[f_0] : x \mapsto f(x + x_*)$ in $\mathcal{E}^n_n$ is $\mathcal{K}$-equivalent to the germ $[s^n_{1^r}] : (x_1, x_2) \mapsto (x_1^{r+1}, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.*

4 The $\mathcal{B}$-$\mathcal{G}$ conditions

Suppose we are given a family of functions (representing vector fields or maps of some dynamical system). In [8, 9], conditions are provided to find the location, i.e. the (x, μ) values, of underlying catastrophes present in this family, without having to reduce the functions to some normal form.

This is achieved using a sequence of matrix determinants, which will be denoted by symbols $\mathcal{B}_{j,i_1...i_j}$ for $j = 1, ..., r$, and $i_1, ..., i_j \in \{1, ..., n\}$, for a catastrophe of codimension r. For readability we will give the conditions for underlying catastrophes in terms of these $\mathcal{B}$ symbols, before defining the functions $\mathcal{B}$ themselves.

Lemma 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x_* \in \mathbb{R}^n$. The germ $[f]$ at $x = x_*$ is **an underlying catastrophe germ** of corank 1 and codimension r if, and only if, the following conditions hold at the point $x = x_*$:*

(a) $0 = f = \mathcal{B}_1 = \mathcal{B}_{2,i_1} = \mathcal{B}_{3,i_1 i_2} = \cdots = \mathcal{B}_{r,i_1 i_2 ... i_{r-1}}$ for all $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$;

(b) $\mathcal{B}_{r+1,i_1 i_2 ... i_r} \neq 0$ for at least one choice of $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$.

The functions $\mathcal{B}_{j,i_1...i_{j-1}}$ are defined in (5) below.

The proof is in [9], where it is shown that each function $\mathcal{B}_{j,k_1...k_{j-1}}$ corresponds to calculation of one of the matrix minors characterizing the rank of ideals in the Thom-Boardman classification of corank 1 (labeled $\Sigma^{1...1}$) singularities.

The $\mathcal{B}$'s themselves are defined as follows. $\mathcal{B}_1$ is simply the determinant of the Jacobian of f. Then for any $i_1 \in \{1, \ldots, n\}$, the quantity $\mathcal{B}_{2,i_1}$ is the determinant of the Jacobian of the function obtained by replacing the i_1-th component of f by $\mathcal{B}_1$, and so on iteratively: for any $j \in \mathbb{N}$, the quantity $\mathcal{B}_{j,i_1...i_{j-1}}$ is defined by the determinant of the Jacobian of the function obtained by replacing the i_{j-1}-th component of f by $\mathcal{B}_{j-1,\,i_1\ldots i_{j-2}}$. In symbols, we write

$$\begin{aligned}
\mathcal{B}_1 &= \left|\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right| \;, \\
\mathcal{B}_{2,\,i_1} &= \left|\frac{\partial(f_1,\ldots,\,f_{i_1-1}\,,\,\mathcal{B}_1\,,\,f_{i_1+1},\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right| \;, \\
\mathcal{B}_{3,\,i_1 i_2} &= \left|\frac{\partial(f_1,\ldots,f_{i_2-1}\,,\,\mathcal{B}_{2,\,i_1}\,,\,f_{i_2+1},\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right| \;, \\
&\vdots \\
\mathcal{B}_{j,\,i_1\ldots i_{i-2}i_{j-1}} &= \left|\frac{\partial(f_1,\ldots,f_{i_{j-1}-1}\,,\,\mathcal{B}_{j-1,i_1\ldots i_{j-2}}\,,\,f_{k_{i-1}+1},\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right| \;.
\end{aligned} \tag{5}$$

Theorem 4.1 then suggests the following theoretical procedure to classify germs of underlying catastrophes given a dynamical system described by a vector field f, or a map Π with displacement function $f = \Pi - x$. At some x where one conjectures a catastrophe to occur:

1. calculate $\mathcal{B}_1$ (the Jacobian determinant), if this is zero then the point at least a fold germ and we continue ...

2. calculate the functions $\mathcal{B}_{2,i_1 i_2}$ for all $i_1, i_2 \in \{1,...,n\}$, if these are not all zero then the point is a fold germ and we are done, if they are all zero then the point is at least a cusp germ and we continue ...

3. calculate the functions $\mathcal{B}_{3,i_1 i_2 i_3}$ for all $i_1, i_2, i_3, \in \{1,...,n\}$, if these are not all zero then the point is a cusp germ and we are done, if they are all zero then the point is at least a swallowtail germ and we continue ...

4. etc., proceeding through each 'level' j until at least one of the $\mathcal{B}_{j,i_1\ldots i_{j-1}}$ is nonzero.

If one is interested in a particular point (x_*, μ_*) (for instance a certain singularity is expected there), then these conditions with Theorem 4.1 allow one to verify that a catastrophe of codimension r occurs there. Now, at a given point (x_*, μ_*), any standard text on bifurcation theory will provide conditions to test for certain bifurcations *at that point* (though often involving complicated calculation simplified by the lemma above).

The inverse problem, however, is much harder, and in standard bifurcation theory is not generally solveable. If one wishes to find where in (x, μ) some conjectured catastrophe occurs, Theorem 4.1 does not provide a practical means to finding it, as there are too many of these $\mathcal{B}_{j,\,i_1\ldots i_{j-1}}$s at each level ($\frac{n^j-1}{n-1}$ for each $j \in \mathbb{N}$ as shown in [9]), to solve for the r different parameters μ.

Fortunately, we can reduce the different permutations of $\mathcal{B}_{j,\,i_1\ldots i_{j-1}}$ at each j to a single choice of the $\{i_1 \ldots i_{j-1}\}$, without loss of generality, resulting in a set of r conditions $0 = \mathcal{B}_1 = \mathcal{B}_2 = \ldots = \mathcal{B}_r$ that can be uniquely solved to find a codimension r catastrophe. We thus provide here the theory to extend the generality of underlying catastrophes proposed in [9].

To do this, the $\mathcal{B}$s must be calculated in coordinates for which the 'subrank' of f is $n-1$, defined as follows.

Definition 4.1. *The subrank of a function $f = (f_1, f_2, \ldots, f_n)$ at a point $x \in \mathbb{R}^n$, denoted by* subrank $f(x)$, *is the least number of dimensions spanned by any choice of $n-1$ of the gradient vectors $\frac{\partial f_1}{\partial x}(x), \ldots, \frac{\partial f_n}{\partial x}(x)$.*

The simplifying result is essentially that, if the subrank of f at a point is equal to $n-1$, then the vanishing of an arbitrary choice of the $\mathcal{B}$'s at each level implies vanishing of all of them (see [9, Corollary 4.2]). This implies the following.

Lemma 4.2. $[f]$ *is an* ***underlying catastrophe germ*** *of corank 1 and codimension* r *if the following conditions hold at the point* $x = x_*$*:*

(a) subrank $f = n-1$

(b) $0 = f = \mathcal{B}_1 = \mathcal{B}_{2,\,i_1} = \mathcal{B}_{3,\,i_1 i_2} = \cdots = \mathcal{B}_{r,\,i_1 i_2 \ldots i_{r-1}}$ *for a choice* $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$*;*

(c) $\mathcal{B}_{r+1,\,i_1 i_2 \ldots i_r} \neq 0$ *for at least one choice of* $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$.

The condition "subrank $f = n-1$" can be interpreted as the system being expressed in a 'good' (generic) coordinate system with respect to the singularity. We can then simplify notation by writing each $\mathcal{B}$ function as some $\mathcal{B}_j := \mathcal{B}_{j,\,i_1 \ldots i_{j-1}}$ for the given choice $i_1, i_2, \ldots, i_r \in \{1, \ldots, n\}$ (and in [8, 9] the choice $i_1 = \ldots = i_{j-1}$ is usually taken for convenience, but different choices may prove more algebraically efficient).

Importantly, this restriction to 'good' coordinates does not reduce generality, as such coordinates exist for any corank 1 singularity.

Lemma 4.3. *Every germ* $[f] \in \mathcal{E}_n^n$ *of a corank* 1 *singularity is* $\mathcal{K}$*-equivalent to a germ whose subrank is* $n-1$.

Proof. Gaussian elimination ensures the existence of an invertible matrix E such that $E \cdot Df(0)$ is in reduced row echelon form. Since $Df(0)$ has corank 1, this means that $E \cdot Df(0)$ has $n-1$ columns equal to the first $n-1$ elements of the canonical basis of $\mathbb{R}^n$, and one column – not necessarily the last one – that is a linear combination of those. In particular, its last row is zero. Hence, a permutation in the ordering of coordinates $\varphi : x_i \mapsto y_i = x_{j(i)}$ can be chosen to rearrange the columns so that

$$E \cdot Df(0) \cdot D\varphi(0) = \left[\begin{array}{ccc|c} & I_{n-1} & & * \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] , \tag{6}$$

where I_{n-1} denotes the $(n-1)$-dimensional identity matrix.

Define

$$M = \left[\begin{array}{ccc|c} & I_{n-1} & & 0 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right] , \tag{7}$$

so that

$$M \cdot E \cdot Df(0) \cdot D\varphi(0) = \left[\begin{array}{ccc|c} & I_{n-1} & & * \\ \hline 1 & \cdots & 1 & * \end{array}\right] . \tag{8}$$

Thus it is easy to see that $[g] = [M \cdot E] \cdot [f] \circ [\varphi]$ has subrank equal to $n-1$. □

Our promised goal of the equations $0 = \mathcal{B}_1 = \ldots = \mathcal{B}_r$ now providing a solvable set of r equations to locate a codimension r catastrophe is not quite complete, however, as there is not yet any guarantee that these are solvable. To ensure solvability requires a second notion from [8], that the family F with germ f is 'full'. We introduce this concept below in Section 4.1.

Note that in moving from the discussion of the $\mathcal{B}$ conditions above, to 'full' families below, we go from considering the germ f to considering the family F. Hence the discussion above makes mention only of the coordinates x and not the parameters μ of the family. A family F may contain one such germ f without possessing enough parameters to fully unfold the catastrophe (less parameters than the codimension of the germ) or its set of parameters, even though sufficient in number, may not be transversal. In these cases F will not be full (see Section 4.1), and it will not be possible to solve for the parameters, even if the $\mathcal{B}$ conditions are satisfied at some point. That does not change the fact that the germ of such a catastrophe appears in that family.

4.1 Full catastrophes

As mentioned above, there are conditions that guarantee it is possible to locate an underlying catastrophe by solving the $\mathcal{B}$ conditions, if a family F is *full*, the definition of which is as follows. As in Theorem 4.1 we specify conditions involving functions $\mathcal{G}_{j,...}$ before giving their definitions below, for readability.

Definition 4.2. *Any local family $F(x,\mu)$ (in good coordinates) at (x_*,μ_*) whose critical germ is an underlying catastrophe germ of corank* 1 *and codimension r is* **full** *if non-degeneracy conditions*

$$\mathcal{G}_{r,\,i_1\ldots i_{r-1}} \neq 0 \tag{9}$$

hold at (x_*,μ_*) for all $i_1, i_2, \ldots, i_{r-1} \in \{1,\ldots,n\}$. The functions $\mathcal{G}_{j,\,i_1\ldots i_{j-1}}$ are defined in (10) *below.*

The $\mathcal{G}_{r,\,i_1\ldots i_{r-1}}$ are a set of extended determinants whose non-vanishing ensures not only that the parameters of the family unfold its singular germ, but also solvability of the conditions stated in item (a) of Theorem 4.1 for arbitrary choices of $i_1, i_2, \ldots, i_r$. These determinants are numerous, but they are straightforward to calculate, and we are only required to verify that they do not vanish. For a family $F = (F_1, \ldots, F_n)$ they are given by:

$$\begin{aligned}
\mathcal{G}_1 &= \left|\frac{\partial(F_1,\ldots,F_n,\,\mathcal{B}_1)}{\partial(x_1,\ldots,x_n,\mu_1)}\right| \,, \\
\mathcal{G}_{2,\,i_1} &= \left|\frac{\partial(F_1,\ldots,F_n,\,\mathcal{B}_1\,,\,\mathcal{B}_{2,i_1})}{\partial(x_1,\ldots,x_n,\mu_1,\mu_2)}\right| \,, \\
\mathcal{G}_{3,\,i_1 i_2} &= \left|\frac{\partial(F_1,\ldots,F_n,\,\mathcal{B}_1\,,\,\mathcal{B}_{2,\,i_1}\,,\,\mathcal{B}_{3,\,i_1 i_2})}{\partial(x_1,\ldots,x_n,\mu_1,\mu_2,\mu_3)}\right| \,, \\
&\;\;\vdots \\
\mathcal{G}_{r,\,i_1\ldots i_{r-1}} &= \left|\frac{\partial(F_1,\ldots,F_n,\mathcal{B}_1\,,\,\mathcal{B}_{2,\,i_1}\,,\ldots,\,\mathcal{B}_{r,\,i_1\ldots i_{r-1}})}{\partial(x_1,\ldots,x_n,\mu_1,\ldots,\mu_r)}\right| \,.
\end{aligned} \tag{10}$$

Any full family is universal, i.e. has exactly as many parameters as needed to completely unfold its singular germ. The more precise statement of this fact is given by the following.

Proposition 4.4. *Any full local family at (x_*,μ_*) whose critical germ is an underlying catastrophe germ of corank 1 and codimension r unfolds that same underlying catastrophe.*

Proof. Let $F_\varepsilon(x,\mu) = F(x,\mu) + \varepsilon H(x,\mu,\varepsilon)$ for $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ be a smooth ε-perturbation of the family F. Whatever the choice made for the sequence $i_1, i_2, \ldots, i_{r-1}$, fullness ensures that

$\mathcal{G}_{r,i_1,i_2,\ldots,i_{r-1}} \neq 0$. Thus, by an application of the Implicit Function Theorem, there are unique smooth functions $x_*(\varepsilon) = \mu_*(\varepsilon)$ such that $x_*(0) = x_*$, $\mu_*(0) = \mu_*$ and

$$F_\varepsilon = \mathcal{B}_1 = \cdots = \mathcal{B}_r = 0 \tag{11}$$

evaluated at $(x_*(\varepsilon), \mu_*(\varepsilon))$. Moreover, since $\mathcal{B}_{r+1} \neq 0$ at $(x_*(0), \mu_*(0))$ when $\varepsilon = 0$, it follows that $\mathcal{B}_{r+1} \neq 0$ for small values of ε.

Hence there is a unique solution of the conditions equivalent to the existence of an underlying catastrophe germ of corank 1 and codimension r. This implies that the family F is stable. Since we are considering the case of corank 1 singularities, a stable family of a given codimension r is necessarily induced by the normal form of an underlying catastrophe provided in (4). Finally, the fact that F has exactly r parameters ensure that it is actually equivalent to this normal form. □

The next result is a converse of Theorem 4.4 stating that any underlying catastrophe admits a system of coordinates in which it is realized by a full family.

Proposition 4.5. *Every family undergoing an underlying catastrophe of corank* 1 *and codimension* r *is* $\mathcal{K}$*-equivalent to a full family undergoing the same underlying catastrophe.*

Proof. It suffices to prove that the family $F(x,\mu) = (\mathcal{U}_{1^r}(x_1,\mu), x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is $\mathcal{K}$-equivalent to a full family exhibiting the same underlying catastrophe. This can be done by defining

$$M = \left[\begin{array}{c|ccc} 1 & 1 & \cdots & 1 \\ \hline 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \end{array}\right], \tag{12}$$

and taking into account the $\mathcal{K}$-equivalent family

$$G(x,\mu) := M \cdot F(x,\mu) = (\mathcal{U}_{1^r}(x_1,\mu) + (1,\ldots,1)\cdot x_2,\ x_2)\ . \tag{13}$$

G is a primary form (see [9]) of the same catastrophe exhibited by F, and those are known to be full families (see Section 4.5 of the same reference). □

Hence, if one is interested in locating underlying catastrophes, it is not only natural to solve for full families, but there is no loss of generality in only looking for full families if the possibility of having to use different coordinate systems is kept in mind. Thus, the notion of full families provides a practical and explicitly solvable method to find any underlying catastrophe of corank 1 and a given codimension r in a system: solve the system $0 = F = \mathcal{B}_1 = \ldots = \mathcal{B}_r$ in (x,μ) for some r, then check for fullness by verifying $\mathcal{G}_{r,i_1 \ldots i_{r-1}} \neq 0$ for all $i_1 \ldots i_{r-1} \in \{1,\ldots,n\}$.

Two important questions naturally emerge from the theory, particularly in relation to the use of different coordinate systems. The first is whether every r-family unfolding an underlying catastrophe of corank 1 and codimension r is full in good coordinates, i.e., coordinates satisfying the subrank condition provided previously. If answered, this question would not only provide a substantial improvement of the practical method outlined above, but also allow us to characterize full families more simply by two properties: a *good coordinate system* and *parameters that completely unfold* its singular germ.

The second is whether the so-called good coordinate systems for which an underlying catastrophe is realized by a full family are 'generic' in some sense, so that random changes of coordinates would most likely allow us to verify fullness for universal families.

4.2 Examples

Example 4.1 (A non-full cusp catastrophe)**.** *The scalar function $F = x_1^3 + \alpha x_1 + \beta$ is the normal form of a cusp catastrophe. However, if we simply embed this in a planar system as*

$$(F_1, F_2) = \left(x_1^3 + \alpha x_1 + \beta,\ x_2\right)\ . \tag{14}$$

then the cusp we obtain is not 'full' according to Definition 4.2, because the coordinates (x_1, x_2) here are not 'good' in the sense given following Theorem 4.2. That is, $F = (F_1, F_2)$ does not have subrank $n - 1 = 1$, it has subrank 0 by Definition 4.1, essentially because the systems F_1 and F_2 are uncoupled. Introducing an arbitrary coupling that does not affect the catastrophe will resolve this, most simply by adding $(kx_2, 0)$ for any $k \neq 0$.

For a bifurcation analysis, then, first note that the equilibria lie at $F = 0$, and rather than solve the cubic equation for x_1, let us instead note that equilibria lie at $x_2 = 0$ and $\beta = -x_1^3 - \alpha x_1$.

Folds then occur where $\mathcal{B}_1 = 3x_1^2 + \alpha = 0$, hence (evaluated at the equilibria), where $(x_1, x_2, \beta) = (\pm\sqrt{-\alpha/3}, 0, \pm 2(-\alpha/3)^{3/2})$. We can consider folds that occur as we vary α, in which case $\mathcal{G}_1 = 2\alpha$, or as we vary β, in which case $\mathcal{G}_1 = \mp 2\sqrt{-3\alpha}$. In either case the fold is full provided $\alpha \neq 0$. Evidently a degeneracy occurs at $\alpha = 0$.

To look for cusps we then calculate

$$\begin{aligned} 0 &= \mathcal{B}_{2,1} = 6x_1 \\ 0 &= \mathcal{B}_{2,2} = -k6x_1\ . \end{aligned} \tag{15}$$

We see that $\mathcal{B}_{2,1}$ and $\mathcal{B}_{2,2}$ both vanish uniquely at $x_1 = 0$ only if $k \neq 0$. If $k = 0$ then F is not full and $\mathcal{B}_{2,1} \equiv 0$ or $\mathcal{B}_{2,2} \equiv 0$ tell us nothing. Instead, with $k \neq 0$, we can arbitrarily choose $\mathcal{B}_2$ to be either $\mathcal{B}_{2,1}$ and $\mathcal{B}_{2,2}$. With either choice, solving the system $0 = F = \mathcal{B}_1 = \mathcal{B}_2$ then gives cusps at $(x_1, x_2, \alpha, \beta) = (0, 0, 0, 0)$. We verify these are valid, full, and non-degenerate, by calculating $\mathcal{B}_3 = 6$, and $\mathcal{G}_{2,1} = -6$, and $\mathcal{G}_{2,2} = 6k$, again confirming these are only all nonzero when the cusp is full for $k \neq 0$.

Example 4.2 (An incomplete catastrophe)**.** *The system*

$$F(x, \mu) = \left(x_1^3 + \mu x_1 + x_2,\ x_2\right)\ , \tag{16}$$

appears similar to the cusp, but lacks sufficient parameters. We can solve for folds as in the previous example. The subrank of F is $n - 1 = 1$, as required to be full. When we look for cusps, we calculate

$$\begin{aligned} 0 &= \mathcal{B}_{2,1} = 6x_1 \\ 0 &= \mathcal{B}_{2,2} = -6x_1\ , \end{aligned} \tag{17}$$

revealing that indeed the system has a cusp germ at $x_1 = 0$. However, to calculate $\mathcal{G}_{2,1}$ and $\mathcal{G}_{2,2}$ we need two parameters, and we have only one. Since we cannot define the $\mathcal{G}_{2,i}$s this cannot be full, hence it is not a cusp catastrophe, merely a one-parameter section through a cusp catastrophe known, of course, as a pitchfork bifurcation.

Example 4.3 (A biological membrane model – butterflies and stars)**.** *Underlying catastrophes permit us to study systems that would be difficult to tackle with standard bifurcation analysis. In*

[1] they are used to determine when certain 'wave-pinning' states can exist in a biological reaction-diffusion equation, expressible in dimensionless form as

$$\frac{\partial}{\partial t}(x_1,x_2) = \frac{\partial^2}{\partial u^2}(x_1,x_2) + (F_1,F_2) \ , \tag{18}$$

where $t \in \mathbb{R}$ is time, $u \in \mathbb{R}$ is a spatial coordinate, and (x_1, x_2) represent membrane potentials. A more detailed description of how this system models the process of cell differentiation can also be found in [1], but, for the purposes of this paper, it suffices to know that we will mostly be interested in homogeneous steady states of (18), i.e., $F_1 = F_2 = 0$.

Hence, our main concern is the activation-inhibition dynamics described by the functions

$$(F_1,F_2) = \left(\rho x_2 + x_1^3 + \alpha x_1 + k x_1 x_2 + \beta,\ \sigma x_1 + x_2^3 + \gamma x_2 + k x_1 x_2 + \delta\right) \ . \tag{19}$$

The zeros of $F = (F_1, F_2)$, which are the homogeneous steady states, undergo catastrophes of up to codimension $r = 6$. Let us briefly summarise the analysis, further details can also be found in [8] and a more complete numerical analysis is given in [1].

The kx_1x_2 term in (20) makes this a non-gradient system, so for $k \neq 0$ this is not reducible to a 1-dimensional form by coordinate transformation, hence the $\mathcal{B}$-$\mathcal{G}$ conditions provide perhaps the only practical means to find its singularities. We will first take the case $k = 0$ as this simplifies the calculations considerably, and the system still exhibits catastrophes up to codimension 4, then we will show that for $k \neq 0$ the catastrophes continue up to codimension 6.

So, setting $k = 0$, we wish first to find the solutions of $F = 0$. Rather than solve this 9^{th} order polynomial in (x_1, x_2), it is more convenient to say the zeroes occur at (β, δ) values given by

$$\beta = -\rho x_2 - x_1^3 - \alpha x_1 \ , \quad \delta = -\sigma x_1 - x_2^3 - \gamma x_2 \ . \tag{20}$$

To find catastrophes we then solve

$$0 = \mathcal{B}_1 = \left|\frac{\partial(F_1,F_2)}{\partial(x_1,x_2)}\right| = (\alpha + 3x^2)(\gamma + 3x_2^2) - \rho\sigma \ , \tag{21}$$

to find that folds occur at (α, β, δ) values given by

$$\alpha = \frac{\rho\sigma}{\gamma+3x_2^2} - 3x_1^2 \ , \quad \beta = 2x_1^3 - \rho x_2 - \frac{\rho\sigma x_1}{\gamma+3x_2^2} \ , \quad \delta = -\sigma x_1 - x_2^3 - \gamma x_2 \ . \tag{22}$$

Solving next

$$0 = \mathcal{B}_2 = \left|\frac{\partial(\mathcal{B}_1,F_2)}{\partial(x_1,x_2)}\right| = 6x_1(\gamma + 3x_2^2)^2 - 6\sigma x_2(\alpha + 3x_1^2) \ , \tag{23}$$

we find that cusps occur where $\gamma = \left(\rho\sigma^2 x_2/x_1\right)^{1/3} - 3x_2^2$, and hence at $(\alpha, \beta, \gamma, \delta)$ values

$$\alpha = \gamma = a(x_1,x_2) \ , \quad \beta = \delta = -\sigma x_1 - x_2^3 - x_2 a(x_1,x_2) \ , \tag{24}$$

in terms of a function

$$a(s,t) := (\rho\sigma^2 x_2/x_1)^{1/3} - 3x_2^2 \ . \tag{25}$$

Solving

$$0 = \mathcal{B}_3 = \left|\frac{\partial(\mathcal{B}_2,F_2)}{\partial(x_1,x_2)}\right| = 6\sigma^2(\alpha + 3x_1^2) + 6(\gamma + 3x_2^2)\left((\gamma + 3x_2^2)^2 - 18\sigma x_1 x_2\right) \ , \tag{26}$$

is slightly trickier, but as suggested in [8], if we let $(u,v) = (x_1/x_2, x_1x_2)$*, then* $\mathcal{B}_3 = 0$ *(along with the preceding conditions) has the solution* $v = (\rho\sigma)^{1/3}(\sigma^{1/3}u^{2/3} + \rho^{1/3}u^{-2/3})/18$*, hence we have swallowtails at* $(\alpha,\beta,\gamma,\delta,v)$ *values given by*

$$\begin{aligned} \alpha = \gamma &= (\rho\sigma^2/u)^{1/3} - 3v/u \;, \\ \beta = \delta &= (v/u)^{1/2}\left(2v/u - \sigma u - (\rho\sigma^2/u)^{1/3}\right) \;, \\ v &= (\rho\sigma)^{1/3}(\sigma^{1/3}u^{2/3} + \rho^{1/3}u^{-2/3})/18 \;. \end{aligned} \tag{27}$$

Lastly let us solve

$$0 = \mathcal{B}_4 = \left|\frac{\partial(\mathcal{B}_3,F_2)}{\partial(x,y)}\right| = 72\sigma\left(15\sigma x_1x_2^2 - 3\gamma^2x_2 - 27x_2^5 + 2\gamma(x_1\sigma - 9x_2^3)\right) \;, \tag{28}$$

which gives butterfly catastrophes at

$$\begin{aligned} (x_1,x_2) &= \pm\tfrac{1}{3}(\rho\sigma)^{1/8}(\rho^{1/4},\sigma^{1/4}) \;, \\ (\alpha,\gamma) &= \tfrac{2}{3}(\rho\sigma)^{1/4}(\rho^{1/2},\sigma^{1/2}) \;, \\ (\beta,\delta) &= \mp\tfrac{16}{27}(\rho\sigma)^{3/8}(\rho^{3/4},\sigma^{3/4}) \;. \end{aligned} \tag{29}$$

For brevity we have not given the non-degeneracy conditions at each stage above, but one may easily confirm that at each level the appropriate $\mathcal{G}_{j,i_1\ldots i_{j-1}}$ *are all nonzero (except where each catastrophe degenerates into one of higher order, e.g. the fold becomes a cusp, etc.). For the butterfly, for instance, they evaluate as* $\mathcal{G}_{4,i_1i_2i_3} = \pm 103680\rho^p\sigma^{8-p}$ *with* $p \in \left[\frac{7}{4},\frac{25}{4}\right]$ *for any* $i_1,i_2,i_3 \in \{1,2\}$*, remaining non-degenerate provided* ρ *and* σ *do not vanish.*

With $k = 0$*, these are the only full catastrophes in the system. If we now let* $k \neq 0$ *we find also wigwam and star catastrophes. Solving* $0 = F = \mathcal{B}_1 = ... = \mathcal{B}_6$*, (omitting the calculations now for brevity), we find a star catastrophe at*

$$x_1 = x_2 = \tfrac{1}{6}k \;, \quad \alpha = \gamma = -\tfrac{1}{2}k^2 \;, \quad \beta = \delta = \tfrac{13}{108}k^3 \;, \quad \rho = \sigma = -\tfrac{5}{12}k^2 \;. \tag{30}$$

This is full for $k \neq 0$*, with* $\mathcal{G}_{6,i_1i_2i_3i_4i_5} = \pm\frac{3^85^27}{2^{15}}k^{29}$ *for any* $i_1,i_2,i_3,i_4,i_5 \in \{1,2\}$*.*

The analysis above describes an intriguing picture of the catastrophe geometry in the gradient case $k = 0$ *as compared to the general case* $k \neq 0$*, beyond that investigated in [8, 1]. This is illustrated schematically in fig. 1, depicting the intersection of these spaces with the higher codimension catastrophe sets in the 7-dimensional space of* $(\alpha,\beta,\gamma,\delta,\sigma,\rho,k)$*. (Numbers in the figure indicate the dimensions of each set, depicted with one dimension less: i.e. a plane is depicted as a line, etc.). The gradient system has a* (ρ,σ)*-parameterised surface of butterflies within the set* $k = 0$*. This is just the intersection set of a volume of* (ρ,σ,k)*-parameterised butterflies of the full system, which becomes singular along a surface of* (ρ,k)*-parameterised wigwams, which in turn becomes singular along a curve of* k*-parameterised star catastrophes.*

Note that none of the catastrophes found above for this system lie at the origin, where F *has catastrophe germ* (x_1^3,x_2^3)*. This catastrophe at the origin is not a corank 1 catastrophe, but has corank 2, so it is beyond our scope here and remains to be studied in future work.*

5 Bifurcations in families of vector fields

The prevailing approach when studying qualitative changes in families of vector fields has been to analyze them in terms of topological equivalence. We present a brief summary of the concepts necessary for such analysis, including the crucial concept of reduction to centre manifold. The notion

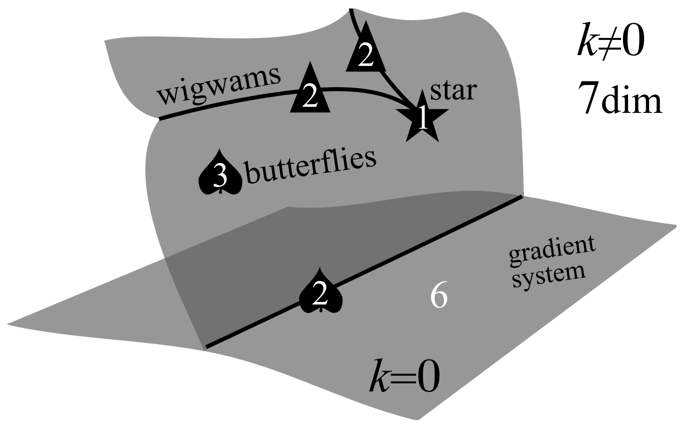

Figure 1: Schematic illustration of the singularity sets with respect to the set $k = 0$ (on which (20) is a gradient system), indicating the star ★, wigwam ▲, and butterfly ♠ catastrophes. White numerals indicate the dimension of each set in the 7-dimensional space of $(\alpha, \beta, \gamma, \delta, \sigma, \rho, k)$.

of versality (with respect to topological equivalence) is also discussed, allowing the introduction of principal families.

5.1 Local families of vector fields and topological equivalence

Definition 5.1. *A germ $[\xi]$ at x_* is said to be the germ of a local homeomorphism if there is one element ξ in the class $[\xi]$ for which:*

(*i*) $\xi(x_*) = x_*$;

(*ii*) ξ *is a local homeomorphism near* x_*.

The set of germs of local homeomorphisms at x_* on $\mathbb{R}^n$ is denoted by $H_n(x_*)$.

Definition 5.2. *Two germs $[f]$ and $[g]$ of vector fields at x_* in $\mathbb{R}^n$ are topologically equivalent if there is a germ $[\xi] \in H_n(x_*)$ of a local homeomorphism that transforms trajectories of $\dot{x} = f(x)$ into trajectories of $\dot{y} = g(y)$ preserving the direction of time.*

Definition 5.3. *Let $\dot{x} = F(x, \mu)$ be a k-parameter smooth family of differential equations in $\mathbb{R}^n$. For any (x_*, μ_*), we say that the germ of F at (x_*, μ_*) is a* ***local k-parameter smooth family of vector fields at*** (x_*, μ_*). *The germ of $x \mapsto F(x, \mu_*)$ is the critical germ of the family.*

Definition 5.4. *Two local k-parameter smooth families $[F]$ and $[G]$ of vector fields at (x_*, μ_*) are topologically equivalent if:*

(*i*) *the germs of $f : x \mapsto F(x, \mu_*)$ and $g : x \mapsto G(x, \mu_*)$ at $x = x_*$ are topologically equivalent, i.e., there is a local homeomorphism in the space of coordinates $[\xi] \in H_n(x_*)$ taking trajectories of $\dot{x} = f(x)$ into trajectories of $\dot{y} = g(y)$ and preserving the direction of time;*

(*ii*) *there are a local homeomorphism in the space of parameters $[h] \in H_k(x_*)$, and a k-parameter unfolding $[\tilde{\Phi}] : (x, \mu) \mapsto (\Phi(x, \mu), \mu)$ of $[\xi]$ in $\mathbb{R}^n$ having a representative Φ such that, for each μ fixed, $x \mapsto \Phi(x, \mu)$ is a homeomorphism that transforms trajectories (in its domain) of $\dot{x} = F(x, \mu)$ into trajectories of $\dot{y} = G(y, h(\mu))$ and preserves the direction of time.*

5.2 Centre manifolds and the Reduction Theorem

Let $[F]$ be a local k-parameter family of vector fields such that:

- $F(x_*, \mu_*) = 0$;
- $DF(x_*, \mu_*)$ has n_c eigenvalues with zero real part.

A centre manifold $\mathcal{W}^c_{[F]}$ of $[F]$ at (x_*, μ_*) is defined as a centre manifold of the system

$$\dot{x} = F(x, \mu), \quad \dot{\mu} = 0, \tag{31}$$

at (x_*, μ_*), where F is any representative of $[F]$.

Let $(z, \mu) \in \mathbb{R}^{n_c} \times \mathbb{R}^k$ be local coordinates for the centre manifold $\mathcal{W}^c_{[F]}$ such that $(z, \mu) = (0, \mu_*)$ corresponds to $(x, \mu) = (x_*, \mu_*)$. The restriction $\dot{z} = F_{\text{center}}(z, \mu)$, $\dot{\mu} = 0$ of (31) to $\mathcal{W}^c_{[F]}$ in these coordinates defines a local k-parameter family $[F_{\text{center}}]$ at $(0, \mu_*)$, named the restriction of $[F]$ to its centre manifold at (x_*, μ_*).

A form of reversal of restriction to centre manifold is provided by the saddle-suspension of a family.

Definition 5.5. *Let $n_s, n_u \in \mathbb{N}$. The (n_s, n_u)-saddle suspension of a family $\dot{x} = Z(x, \mu)$, $x \in U \subset \mathbb{R}^{n_c}$, of differential equations is the family*

$$\dot{x}_c = Z(x_c, \mu), \quad \dot{x}_s = -x_s, \quad \dot{x}_u = x_u \ , \tag{32}$$

where $x_c \in U \subset \mathbb{R}^{n_c}$, $x_s \in \mathbb{R}^{n_s}$, $x_u \in \mathbb{R}^{n_u}$.

The (n_s, n_u)-saddle suspension of a local family $[Z]$ is the local family given by the (n_s, n_u)-saddle suspension of $\dot{x} = Z(x, \mu)$ for any representative Z of $[Z]$.

The restriction of a family to its centre manifold captures all the non-trivial behavior. This is stated precisely in the following classic theorem (see [2, 16]).

Theorem 5.1 (Reduction Theorem). *Let $[F]$ be local k-parameter family at (x_*, μ_*) satisfying:*

- *$F(x_*, \mu_*) = 0$;*
- *$DF(x_*, \mu_*)$ has n_c eigenvalues with zero real part, n_s eigenvalues with negative real part, and n_u eigenvalues with positive real part.*

Then $[F]$ is topologically equivalent to the local (n_s, n_u)-saddle suspension of the restriction $[F_{center}]$ of $[F]$ to its centre manifold at (x_*, μ_*).

5.3 Versality and principal families

In this section we define versal and principal families, and recall a classic result concerning singularities whose centre manifolds are one-dimensional.

Definition 5.6. *A local l-parameter family of vector fields $[G]$ is said to be induced by the local k-parameter family $[F]$ if there is a continuous germ $[h] : (\mathbb{R}^l, 0) \to (\mathbb{R}^k, 0)$ such that $G(x, \eta) = F(x, h(\eta))$.*

Definition 5.7. *A local k-parameter family $[F]$ having $[f]$ as its critical germ is said to be a topologically versal if any other local family having the same critical germ is induced by $[F]$.*

The idea that versal families carry all the information about how a singularity can unfold under any change of parameters leads us to the concept of principal families. For convenience, we start with a one-dimensional differential equation $\dot{x} = f(x)$ having an equilibrium at $x = x_*$.

If this equilibrium is hyperbolic, that is, if $f'(x_*) \neq 0$, it is well-known that it persists, along with its local phase portrait, under smooth perturbations of f. Hence, no significant changes appear for any unfolding, and the trivial family $F(x, \mu) = f(x)$ is versal.

We assume that $f'(x_*) = 0$. In that case, it has been proved that the order of the degeneracy of f precisely defines the least number of parameters that a family must have to versally unfold this singularity (see [2, Section 3.1]). More specifically, if $r \in \mathbb{N}^*$ is such that $f^{(i)}(x_*) = 0$ for $i \in \{0, \ldots, r\}$ but $f^{(r+1)}(x_*) = a \neq 0$, then the minimum number of parameters necessary for a topologically versal unfolding of this singularity is r and one such unfolding is given by

$$\dot{x} = \mathrm{sign}(a)\, x^{r+1} + \mu_{r-1} x^{r-1} + \ldots + \mu_0 \; . \tag{33}$$

These families are known as 'topological normal forms' or 'principal families' (see e.g. [2]). Conventionally, $\mu = 0$ corresponds to the original degenerate system $\dot{x} = f(x)$.

5.4 Bifurcations of centre dimension 1

The one-dimensional principal families presented in the last section are, except for the sign of the leading term, identical to the underlying catastrophes defined in Section 3.4. Motivated by this parallelism, if a local r-parameter family is topologically equivalent to the above-mentioned r-parameter principal family, we say that it undergoes a:

- *fold* bifurcation, if $k = 1$;
- *cusp* bifurcation, if $k = 2$;
- *swallowtail* bifurcation, if $k = 3$;
- *butterfly* bifurcation, if $k = 4$;
- etc. through *wigam*, *star* bifurcations, and so on (for nomenclature see e.g. [15]).

We can now use the term 'bifurcation' over 'underlying catastrophe' since we have that the family F is *topologically* (not just $\mathcal{K}$) equivalent to the principal family (33).

More generally, we can expand this classification to families of vector fields in any number of dimensions directly by saying, for example, that it undergoes a fold bifurcation if it is topologically equivalent to a saddle suspension of a one-dimensional family that undergoes a fold bifurcation as just defined.

Because of the Reduction Theorem (Theorem 5.1), this provides a class for any generic bifurcation that can occur in a family whose critical germ has a one-dimensional centre manifold, which we can refer to as **bifurcations of centre dimension 1**.

6 Bifurcations in families of maps

We can extend these concepts to maps (here strictly meaning diffeomorphisms) that define difference equations, hence extending them from continuous time dynamics (e.g. the vector fields of ordinary differential equations) to discrete time (e.g. Poincare or stroboscopic maps). The adaption from vector fields to diffeomorphisms requires small but quite profound moderations that we present in this section.

Definition 6.1. *Let $\Pi(x, \mu)$ be a k-parameter family of maps in $\mathbb{R}^n$. We define the germ of Π at (x_*, μ_*) as a **local k-parameter family of maps.***

The most important difference here is that the family whose catastrophes of interest exist in the so-called displacement function, which we define as follows.

Definition 6.2. *Let $[\Pi]$ be a local k-parameter family of maps in $\mathbb{R}^n$ at $(x_*, \mu*)$. The local family of displacement functions associated to $[\Pi]$ is the local k-parameter family of maps at (x_*, μ_*) given by $(x, \mu) \mapsto \Pi(x, \mu) - x$.*

6.1 Local families of maps and topological conjugacy

Definition 6.3. *Two germs $[f]$ and $[g]$ of maps at x_* in $\mathbb{R}^n$ are topologically conjugate if there is the germ of a homeomorphism $[\xi] \in H_n(x_*)$ such that*

$$[\xi] \circ [f] = [g] \circ [\xi] \ .$$

Definition 6.4. *Two local k-parameter smooth families $[F]$ and $[G]$ of maps are topologically conjugate if:*

(i) the germs of $f : x \mapsto F(x, 0)$ and $g : x \mapsto G(x, 0)$ at the origin are topologically conjugate;

(ii) letting $[\tilde{F}] : (x, \mu) \mapsto (F(x, \mu), \mu)$ and $[\tilde{G}] : (x, \mu) \mapsto (G(x, \mu), \mu)$, there are all of the following:

- *a local homeomorphism in the space of parameters $[h] \in H_k$,*
- *a local homeomorphism in the space of coordinates $[\xi] \in H_n$, and*
- *a k-parameter unfolding $[\tilde{\Phi}] : (x, \mu) \mapsto (\Phi(x, \mu), \mu)$ of $[\xi]$ in $\mathbb{R}^n$ such that*

$$[\tilde{\Phi}] \circ \Big([h]^*[\tilde{F}]\Big) = [\tilde{G}] \circ [\tilde{\Phi}].$$

There are well-known analogous concepts of centre manifolds for discrete-time dynamical systems, and also a corresponding Reduction Theorem. Those can be found, for instance, in [4] or [11], and will be omitted here for succinctness.

There are no significant differences for the notion of versality of families of maps, and the corresponding principal families are given by those of vector fields added to the identity map, i.e.

$$x \mapsto x + \mathrm{sign}(a)\, x^{k+1} + \mu_{k-1} x^{k-1} + \ldots + \mu_0 \ . \tag{34}$$

Such families also allow the definition of the fold, cusp, swallowtail, butterfly, etc., bifurcation for maps. For example, the principal families corresponding to the fold are $(x, \mu) \mapsto \pm x^2 + x + \mu$. These definitions agree, when overlapping, with the usual naming provided in classic references on the subject (see [11, Theorems 4.2 and 9.1]).

7 Minimal topologically stable local families

Minimal topologically stable families are defined by a set of conditions concerning reduction to a centre manifold under which topological equivalence to a saddle suspension of a principal family is ensured. They can be regarded as a more explicit definition of families undergoing a specific bifurcation. As such, they are an important tool in the proofs of the main results of this paper.

We not only present their definitions, which are slightly different for vector fields and maps, but also prove the above-mentioned results ensuring topological equivalence to principal families.

7.1 Minimal families for vector fields

Definition 7.1. *A k-parameter* ***local family of*** n***-dimensional vector fields*** *$F(x,\mu)$ at (x_*,μ_*) is said to be a* ***centre dimension 1 minimal topologically stable local*** k***-family of vector fields at*** *(x_*,μ_*) if $DF(x_*,\mu_*)$ has exactly one simple eigenvalue equal to zero, all others having non-zero real parts, and the restriction of*

$$\dot{x} = F(x,\mu), \qquad \dot{\mu} = 0 \ , \tag{35}$$

to its centre manifold at (x_*,μ_*) in local coordinates $(y,\mu) \in \mathbb{R}\times\mathbb{R}^k$ for which $(0,\mu_*) \in \mathbb{R}\times\mathbb{R}^k$ corresponds to $(x_*,\mu_*) \in \mathbb{R}^n\times\mathbb{R}^k$, given by

$$\dot{y} = g(y,\mu), \qquad \dot{\mu} = 0 \ , \tag{36}$$

satisfies:

(V.I$_k$) $g(0,\mu_*) = \frac{\partial g}{\partial y}(0,\mu_*) = \cdots = \frac{\partial^k g}{\partial y^k}(0,\mu_*) = 0$;

(V.II$_k$) *$(0,\mu_*)$ is a regular point for the function* $G : (y,\mu) \mapsto \left(g(y,\mu), \frac{\partial g}{\partial y}(y,\mu), \ldots, \frac{\partial^k g}{\partial y^k}(y,\mu)\right)$;

(V.III$_k$) $\frac{\partial^{k+1} g}{\partial y^{k+1}}(0,\mu_*) \neq 0$.

Theorem 7.1. *Every centre dimension 1 minimal topologically stable local k-family of vector fields is topologically equivalent to a saddle suspension of one of the k-parameter principal families (see* (33) *for definition) of one-dimensional vector fields.*

Proof. Without loss of generality, assume that the k-parameter smooth family of vector fields $[F]$ is a minimal topologically stable k-family at $(0,0)$ whose restriction to the centre manifold is $[g]$. By Lemma A.1, it follows that $[g]$ is $\mathcal{K}$-equivalent to $[\mathcal{U}_{1^k}]$. Hence, there are $[h] \in L_k$ and smooth unfoldings $[\tilde{a}]$ of $[a_0] \in \mathrm{GL}_1(\mathcal{E}_1)$ and $[\tilde{\Phi}]$ of $[\phi] \in L_1$ such that $[\tilde{g}] = [\tilde{a}] \cdot \left([h]^*[\tilde{\mathcal{U}}_{1^k}]\right) \circ [\tilde{\Phi}]$. More simply, there are $a(y,\mu) \in \mathbb{R}$ and $\Phi(y,\mu)$ with $a(y,0) = a_0 \neq 0$ and $\Phi(y,0) = y$ such that

$$g(y,\mu) = a(y,\mu)\left(\left(\Phi(y,\mu)\right)^{k+1} + h_{k-1}(\mu)\left(\Phi(y,\mu)\right)^{k-1} + \ldots + h_0(\mu)\right) \tag{37}$$

holds near $(0,0)$.

Let $[g_0] : y \mapsto g(y,0)$ and observe that $[a_0]$ and $[\phi]$ can be any germs for which $[g_0] = [a_0]\cdot[y^{k+1}]\circ[\phi]$. In particular, considering the hypotheses, we obtain by Taylor's theorem that $g_0(y) = y^{k+1}r(y)$, where $r(0) \neq 0$. Hence,

$$g_0(y) = \mathrm{sign}\left(r(0)\right)\left(y\,|r(0)|^{\frac{1}{k+1}}\right)^{k+1} \ , \tag{38}$$

and, by defining $\sigma = \text{sign}(r(0))$, we can choose $a_0(y) = \sigma$ and $\phi(y) = y\,|r(0)|^{\frac{1}{k+1}}$. Thus, we are justified in assuming $a_0(y) = \sigma$ and $\phi'(0) > 0$.

For each fixed small value of μ, let $\{x_1, \ldots, x_l\}$, where $l \leq k+1$ generally depends on the choice of μ, be the real roots (with no repetition) of

$$P_h(x,\mu) := x^{k+1} + h_{k-1}(\mu)x^{k-1} + \ldots + h_0(\mu) = 0 \; . \tag{39}$$

Assume they are indexed so that $x_1 < x_2 < \ldots < x_l$. Since μ is small, and considering that $h(0) = 0$ yields the polynomial x^{k+1}, which has zero as root with multiplicity $k+1$, it follows that all the x_i are also small. Hence, for sufficiently small values of μ, we can safely apply the function ϕ^{-1} to the x_i.

Define

$$y_i := \phi^{-1}(x_i),\; i \in \{1, 2, \ldots, l\} \; . \tag{40}$$

Considering that $\phi'(0) > 0$ implies that $(\phi^{-1})'(0) > 0$, it follows that $y_1 < \ldots < y_l$. Let $\Psi_\mu(x)$ be a strictly increasing bijective function on the real line such that $\Psi_\mu(x_i) = y_i$ for all $i \in \{1, 2, \ldots, l\}$. It is then a homeomorphism of $\mathbb{R}$ onto itself.

We will prove that Ψ_μ takes trajectories of $\dot{y} = g(y,\mu)$ into trajectories of $\dot{x} = \sigma P_h(x,\mu)$. Considering that those are vector fields on the real line and that Ψ_μ associates equilibria to equilibria bijectively, all that remains to be done is proving that it maps stable, unstable-stable, stable-unstable, and unstable equilibria onto, respectively, stable, unstable-stable, stable-unstable, and unstable equilibria. Letting m_i denote the multiplicity of the root x_i of $P_h(x,\mu)$, this is a consequence of the claim

$$\text{sign}\left(\frac{\partial^j g}{\partial y^j}(y_i,\mu)\right) = \sigma\;\text{sign}\left(\frac{\partial^j P_h}{\partial x^j}(x_i,\mu)\right),\; \text{for all } i \in \{1, \ldots, l\} \text{ and all } j \in \{0, \ldots, m_i\} \; , \tag{41}$$

which we prove as follows.

Let $i \in \{1, \ldots, l\}$ and $l \in \mathbb{N}$ be given. Since m_i is the multiplicity of the root x_i of $P_h(x,\mu)$, then there is a polynomial q such that $q(x_i) \neq 0$ and

$$P_h(x,\mu) = (x - x_i)^{m_i} q(x) \; . \tag{42}$$

In particular, it is easy to see that

$$\text{sign}\left(\frac{\partial^j P_h}{\partial x^j}(x_i,\mu)\right) = \begin{cases} 0, \text{ if } j \in \{0, \ldots, m_i - 1\} \; , \\ \text{sign}\, q(x_i) \text{ if } j = m_i \; . \end{cases} \tag{43}$$

Since $g(y,\mu) = a(y,\mu)\, P_h\left(\Phi(y,\mu),\mu\right)$ and $x_i = \Phi(y_i,\mu)$, it also follows that

$$g(y,\mu) = a(y,\mu)\left(\Phi(y,\mu) - \Phi(y_i,\mu)\right)^{m_i} q(\Phi(y,\mu)) \; . \tag{44}$$

Since $a(y,0) = \sigma \neq 0$, by taking μ sufficiently small we can ensure that $\text{sign}(a(y,\mu)) = \sigma$, so that it is easy to verify that

$$\text{sign}\left(\frac{\partial^j g}{\partial y^j}(y_i,\mu)\right) = \begin{cases} 0, \text{ if } j \in \{0, \ldots, m_i - 1\}, \\ \sigma\,\text{sign}\, q(x_i) \text{ if } j = m_i, \end{cases} \tag{45}$$

concluding the claim. We have thus proved that the families $\dot{y} = g(y,\mu)$ and $\dot{x} = \sigma P_h(x,\mu)$ are topologically equivalent (isomorphic, in fact).

To finalise the proof, we will show that $\dot{x} = \sigma P_h(x,\mu)$ is topologically equivalent to the principal family

$$\dot{x} = \mathcal{P}_\sigma(x,\mu) = \sigma x^{k+1} + \mu_{k-1}x^{k-1} + \ldots + \mu_0 \,. \tag{46}$$

If $\sigma = 1$, this follows directly by observing that $P_h(x,\mu) = \mathcal{P}_\sigma(x,h(\mu))$, so that $[P_h] = [h]^*[\mathcal{P}_\sigma]$. If $\sigma = -1$, however, it suffices to observe that $[-h]$ is also the germ of a diffeomorphism and that $[P_h] = [-h]^*[\mathcal{P}_\sigma]$. □

7.2 Minimal families for maps

Definition 7.2. *A k-parameter* ***local family of n-dimensional maps*** *Π at (x_*,μ_*) is said to be a* ***centre dimension 1 minimal topologically stable local k-family of maps at*** *(x_*,μ_*) if $D\Pi(x_*,\mu_*)$ has one simple eigenvalue equal to 1, all others not lying on the unit circle, and the restriction $(t,\mu) \mapsto (g(t,\mu),\mu)$ of the extended map $\Pi_e : (x,\mu) \mapsto (\Pi(x,\mu),\mu)$ to a smooth centre manifold at (x_*,μ_*), in local coordinates $(t,\mu) \in \mathbb{R}\times\mathbb{R}^k$ for which $(0,\mu_*)$ corresponds to (x_*,μ_*), satisfies:*

(M.I$_k$) $g(0,\mu_*) = 0$, $\frac{\partial g}{\partial t}(0,\mu_*) = 1$, *and* $\frac{\partial^2 g}{\partial t^2}(0,0) = \cdots = \frac{\partial^k g}{\partial t^k}(0,\mu_*) = 0$;

(M.II$_k$) $(0,\mu_*)$ *is a regular point for the function* $G : (t,\mu) \mapsto \Big(g(t,\mu) - t, \frac{\partial g}{\partial t}(t,\mu), \ldots, \frac{\partial^k g}{\partial t^k}(t,\mu)\Big)$;

(M.III$_k$) $\frac{\partial^{k+1} g}{\partial t^{k+1}}(0,\mu_*) \neq 0$.

Theorem 7.2. *Every centre dimension 1 minimal topologically stable local k-family of maps is topologically equivalent to a saddle suspension of one of the k-parameter principal families (see* (34) *for definition) of one-dimensional maps.*

Proof. Assume, without loss of generality, that $(x_*,\mu_*) = (0,0)$. We follow the same steps presented in Theorem 7.1, now with the germ of the function $\delta(t,\mu) := g(t,\mu) - t$, which also satisfies items (V.I$_k$) to (V.III$_k$), obtaining, for each small value of μ, the sets $\{x_1,\ldots,x_l\}$ and $\{y_1,\ldots,y_l\}$ of roots of $t \mapsto P_h(t,\mu)$ and $t \mapsto \delta(y,\mu)$, respectively. It is then easy to see that the points x_i and y_i are the fixed points of $t \mapsto t + P_h(t,\mu)$ and $t \mapsto g(t,\mu)$, respectively. Moreover, the same arguments applied in Theorem 7.1 regarding the derivatives ensure that the stability of corresponding fixed points is preserved.

Since $\frac{\partial g}{\partial t}(0,0) = 1$, there are $\bar{t} > 0$ and $\overline{\mu} > 0$ such that $\frac{\partial g}{\partial t}(t,\mu) > \frac{1}{2}$ for all $(t,\mu) \in [-\bar{t},\bar{t}] \times [-\overline{\mu},\overline{\mu}]$. Hence, for any fixed $\mu \in [-\overline{\mu},\overline{\mu}]$, the map $g_\mu : t \mapsto g(t,\mu)$ is strictly increasing on $[-\bar{t},\bar{t}]$. Proceeding similarly, we can assume that the same holds for $(t,\mu) \mapsto t + P_h(t,\mu)$.

We start by defining the conjugating homeomorphism Ψ_μ, whose domain must be $(-\bar{t},\bar{t})$, on the fixed points $\{x_1,\ldots,x_l\}$ of $t \mapsto t + P_h(t,\mu)$, which we can assume to be in this domain for μ small. In fact, we set $\Psi_\mu(x_i) = y_i$ for all $i \in \{1,2,\ldots,l\}$.

Let us consider more carefully what should happen in between two consecutive fixed points x_i and x_{i+1}. Monotonicity and continuity imply that $t \mapsto t + P_h(t,\mu)$ must take (x_i,x_{i+1}) onto itself. Also, since there are no fixed points in this interval, we must have that either $t + P_h(t,\mu) > t$ or $t + P_h(t,\mu) < t$ holds everywhere along it, depending on the stability properties of x_i and x_{i+1}. Without loss of generality, assume that $t + P_h(t,\mu) > t$ holds in (x_i,x_{i+1}).

Then the exact same argument holds for the function g_μ on (y_i, y_{i+1}), and since y_i and y_{i+1} have the same stability properties as x_i and x_{i+1}, we also have $g_\mu(t) > t$ everywhere along this interval.

Now, take any point $t_0 \in (x_i, x_{i+1})$ and define its orbit under $t \mapsto t + P_h(t, \mu)$ to be $\{t_i : i \in \mathbb{Z}\} \subset (x_i, x_{i+1})$. Similarly, take $\{s_i : i \in \mathbb{Z}\} \subset (y_i, y_{i+1})$ to be any orbit under g_μ. Define

$$\Psi_\mu(t) = s_0 + \frac{(t - t_0)}{(t_1 - t_0)}(s_1 - s_0) \tag{47}$$

on $[t_0, t_1]$, which is continuous and strictly increasing. Observe that monotonicity implies that $[t_0, t_1]$ is mapped bijectively onto $[t_1, t_2]$ by $t \mapsto t + P_h(t, \mu)$. Thus, we are now able to define Ψ_μ on $[t_1, t_2]$ by setting

$$\Psi_\mu(t + P_h(t)) = g_\mu(\Psi_\mu(t)) \tag{48}$$

for all $t \in [t_0, t_1]$.

By iterating this argument forward and backwards along the orbit of t_0, we are able to define Ψ_μ on (x_i, x_{i+1}) so that it satisfies the conjugacy relation required by topological equivalence of maps. Finally, the argument has to be repeated on each interval between the fixed points to obtain the homeomorphism Ψ_μ. □

8 Proof of Theorems 2.1 and 2.2: catastrophes of centre dimension 1 characterise bifurcations

In proving the main theorems here, we will differ from the format in previous sections, presenting the case of maps first, since it requires a slightly more involved argument, and then presenting the corresponding result for vector fields, which becomes simpler after Theorem 2.1 in Section 8.2. For convenience, we assume throughout this section that $(x_*, \mu_*) = (0, 0)$ without loss of generality.

8.1 Proof of Theorem 2.2

Two different results are presented here, namely Lemmas 8.1 and 8.2. They imply that any family satisfying the hypotheses given in Theorem 2.2 is a minimally topologically stable family, so that the conclusion of this theorem itself follows from an application of Theorem 7.2.

Lemma 8.1. *Suppose that $[f]$ is the germ of an underlying catastrophe of corank 1 and codimension k at $x = 0$, i.e. there are $[M] \in GL_n(\mathcal{E}_n)$ and $[\phi] \in L_n$ such that $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$. Let $c \in \mathbb{R}^*$ be given and suppose that the derivative of the map $\Pi_0 : x \mapsto x + c\, f(x)$ has exactly one simple eigenvalue equal to 1 at $x = 0$, all others not lying on the unit circle. Define $t \mapsto g(t)$ to be the restriction of this map to its 1-dimensional centre manifold at $x = 0$ in local coordinates such that $t = 0$ corresponds to $x = 0$. Then the following hold:*

(i) $g'(0) = 1$;

(ii) $g^{(i)}(0) = 0$ *for all* $i \in \{2, \ldots, k\}$;

(iii) if $t \mapsto \alpha(t)$ is a regular parametrization of a smooth 1-dimensional centre manifold of Π_0 at $x = 0$, there is a polynomial $p(t) = a_1 t + \ldots + a_k \frac{t^k}{k!}$ such that $a_1 \neq 0$ and the curve $\gamma(t) := \phi^{-1}(p(t), 0)$ has the same k-jet at 0 as α;

(*iv*) $g^{(k+1)}(0) \neq 0$.

Proof. Since $D\Pi_0(0)$ has only one simple eigenvalue equal to 1 on the unit circle, it follows that its centre manifold $\mathcal{M}$ is 1-dimensional and can be written in the form $\{\alpha(t) \in \mathbb{R}^n : t \in (-t_0, t_0)\}$, where $t_0 > 0$, $\alpha(0) = 0$, and $\alpha'(0) \neq 0$. Hence, a right eigenvector v of $D\Pi_0(0)$ associated to 1 has to be a multiple of the derivative of α at 0, i.e., there is $b \neq 0$ such that

$$\alpha'(0) = b\, v\ . \tag{49}$$

Note that v is the right eigenvector of $Df(0)$ associated to a simple zero eigenvalue, because of the identity $\Pi_0(x) = x + cf(x)$. We also define w^T to be a corresponding left eigenvector associated to zero.

Let $g(t)$ denote the restriction of Π_0 to $\mathcal{M}$ in local coordinates. Then it is clear that

$$\alpha(g(t)) = \Pi_0(\alpha(t)) = \alpha(t) + cf(\alpha(t))\ . \tag{50}$$

Differentiating once with respect to t at 0, we obtain

$$\alpha'(0)g'(0) = D\Pi_0(0)\, \alpha'(0) = \alpha'(0) \tag{51}$$

because $\alpha'(0)$ is an eigenvector of $D\Pi_0(0)$ associated to 1. Hence, it follows that

$$g'(0) = 1\ , \tag{52}$$

proving item (*i*).

We will prove item (*ii*) by induction. More precisely, we claim that two different statements hold for $N \in \{2, \ldots, k\}$, to wit:

(*a*) $g^{(N)}(0) = 0$;

(*b*) there is a polynomial

$$p_N(t) = a_1 t + a_2 \frac{t^2}{2} + \ldots + a_N \frac{t^N}{N!} \tag{53}$$

with $a_1 \neq 0$ such that the curve $\gamma_N(t) := \phi^{-1}(p_N(t), 0)$ satisfies $\gamma_N^{(j)}(0) = \alpha^{(j)}(0)$ for all $j \in \{0, 1, \ldots, N\}$.

We start with the base case $i = 2$. By hypothesis, $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$, and we assume henceforth that representatives M of $[M]$ and ϕ of $[\phi]$ are chosen. Define $\gamma_0(t) := \phi^{-1}(t, 0)$. By Lemma A.3, it follows that

$$\frac{\partial f(\gamma_0(t))}{\partial t}\Big|_{t=0} = Df(0)\, \gamma_0'(0) = 0\ . \tag{54}$$

Thus, considering that $\Pi_0(x) = x + cf(x)$, it follows that $\gamma_0'(0)$, which does not vanish on account of its definition, is an eigenvector of $D\Pi_0(0)$ associated to the eigenvalue 1. Since this eigenvalue is simple by hypothesis and $\alpha'(0)$ is also an eigenvector associated to it, we conclude that there is $a_1 \in \mathbb{R}^*$ such that

$$\alpha'(0) = a_1 \gamma_0'(0)\ . \tag{55}$$

Now, taking the second derivative of (50) with respect to t at 0 and considering (52), we obtain

$$\alpha'(0)g''(0) = cD^2f(0)\,\big(\alpha'(0), \alpha'(0)\big) + cDf(0)\alpha''(0)\ , \tag{56}$$

which, when multiplied on the left by w^T, yields

$$b\,w^T v g''(0) = cw^T D^2 f(0)\,\big(\alpha'(0), \alpha'(0)\big)\ . \tag{57}$$

We can now define

$$\gamma_1(t) := \gamma_0\,(a_1 t) = \phi^{-1}\,(a_1 t, 0)\ . \tag{58}$$

Then considering (55), we obtain $\gamma_1'(0) = a_1\gamma_0'(0) = \alpha'(0)$. Another application of Lemma A.3 ensures that

$$\frac{\partial^2 f(\gamma_1(t))}{\partial t^2}\Big|_{t=0} = D^2 f(0)(\gamma_1'(0), \gamma_1'(0)) + Df(0)\gamma_1''(0) = 0\ . \tag{59}$$

Taking the product on the left by w^T, it follows that

$$w^T D^2 f(0)(\alpha'(0), \alpha'(0)) = 0\ , \tag{60}$$

so that (57) combined with Lemma A.2 implies that

$$g''(0) = 0\ . \tag{61}$$

Having proved that, we subtract the product of c by (59) from (56), obtaining

$$Df(0)\,\big(\alpha''(0) - \gamma_1''(0)\big) = 0\ , \tag{62}$$

so that

$$\alpha''(0) - \gamma_1''(0) \in \ker\,(Df(0)) = \langle v\rangle = \langle \gamma_0'(0)\rangle\ . \tag{63}$$

Hence, there is $a_2 \in \mathbb{R}$ such that

$$\alpha''(0) = \gamma_1''(0) + a_2\gamma_0'(0)\ . \tag{64}$$

Define

$$\gamma_2(t) := \phi^{-1}\left(a_1 t + a_2\frac{t^2}{2}, 0\right) \tag{65}$$

and observe that $\gamma_2(0) = 0 = \gamma_1(0)$, $\gamma_2'(0) = a_1\frac{\partial\phi^{-1}}{\partial x_1}(0,0) = \gamma_1'(0)$, and

$$\gamma_2''(0) = a_1^2\frac{\partial^2\phi^{-1}}{\partial x_1^2}(0,0) + a_2\frac{\partial\phi^{-1}}{\partial x_1}(0,0) = \gamma_1''(0) + a_2\gamma_0'(0) = \alpha''(0)\ . \tag{66}$$

This concludes the verification of the base case of the induction.

Let a natural number N such that $2 < N \le k$ be given. Assume, as induction hypothesis, that $g^{(j)}(0) = 0$ and that

$$\gamma_{N-1}(t) = \phi^{-1}\left(a_1 t + a_2\frac{t^2}{2} + \cdots + a_{N-1}\frac{t^{N-1}}{(N-1)!}, 0\right) \tag{67}$$

satisfies $\gamma_{N-1}^{(j)}(0) = \alpha^{(j)}(0)$ for all $j \in \{0, 1, \ldots, N-1\}$.

We take the N-th derivative of (50) with respect to t at $(0,0)$, obtaining

$$\alpha'(0)g^{(N)}(0) = c\frac{\partial^N}{\partial t^N} f(\alpha(t))\Big|_{t=0} . \tag{68}$$

By Faà di Bruno's formula (see [10] for statement) applied to the right-hand side, it follows that

$$\alpha'(0)g^{(N)}(0) = c\sum_{j=0}^{N} D^j f(0)\, B_{N,j}\left(\alpha'(0), \ldots, \alpha^{(N-j+1)}(0)\right) , \tag{69}$$

where $B_{N,j}$ are Bell polynomials (also given in [10]). Considering that the induction hypothesis guarantees that the $(N-1)$-jet of $\alpha(t)$ and $\gamma_{N-1}(t)$ at $t = 0$ coincide, we obtain

$$\alpha'(0)g^{(N)}(0) = c\sum_{j=2}^{N} D^j f(0)\, B_{N,j}\left(\gamma'_{N-1}(0), \ldots, \gamma_{N-1}^{(N-j+1)}(0)\right) + cDf(0)\alpha^{(N)}(0) . \tag{70}$$

In parallel, since $N \leq k$, Lemma A.3 yields

$$0 = \frac{\partial^N}{\partial t^N} f(\gamma_{N-1}(t))\Big|_{t=0} = \sum_{j=2}^{N} D^j f(0)\, B_{N,j}\left(\gamma'_{N-1}(0), \ldots, \gamma_{N-1}^{(N-j+1)}(0)\right) + Df(0)\gamma_{N-1}^{(N)}(0) . \tag{71}$$

Hence, combining (70) and (71), we obtain

$$\alpha'(0)g^{(N)}(0) = cDf(0)\left(\alpha^{(N)}(0) - \gamma_{N-1}^{(N)}(0)\right) = 0 . \tag{72}$$

Multiplying on the left by w^T and considering Lemma A.2, it follows that $g^{(N)}(0) = 0$, and proving the first statement of the induction.

The fact $g^{(N)}(0) = 0$ itself being substituted back into (72) yields

$$\alpha^{(N)}(0) - \gamma_{N-1}^{(N)}(0) \in \ker\left(Df(0)\right) = \langle v \rangle = \langle \gamma'_0(0) \rangle , \tag{73}$$

so that there is $a_N \in \mathbb{R}$ such that

$$\alpha^{(N)}(0) = \gamma_{N-1}^{(N)}(0) + a_N\gamma'_0(0) = \gamma_{N-1}^{(N)}(0) + a_N\frac{\partial \phi^{-1}}{\partial x_1}(0,0) . \tag{74}$$

Define

$$\gamma_N(t) := \phi^{-1}\left(a_1 t + \ldots + a_N\frac{t^N}{N!},\ 0\right) . \tag{75}$$

By Faà di Bruno's formula, if $i \in \{1, 2\ldots, N-1\}$, it follows that

$$\gamma_N^{(i)}(0) = \sum_{j=0}^{i} \frac{\partial^j \phi^{-1}}{\partial x_1^j}(0,0)\, B_{N,j}\left(a_1, \ldots, a_{i-j+1}\right) = \gamma_{N-1}^{(i)}(0) . \tag{76}$$

The same formula yields also for $i = N$,

$$\gamma_N^{(N)}(0) = \sum_{j=0}^{N} \frac{\partial^j \phi^{-1}}{\partial x_1^j}(0,0)\, B_{N,j}\,(a_1, \ldots, a_{i-j+1}) = \gamma_{N-1}^{(N)}(0) + \frac{\partial \phi^{-1}}{\partial x_1}(0,0)\, a_N = \alpha^{(N)}(0)\;, \tag{77}$$

concluding the induction argument and proving item (ii). Also, it is clear that the polynomial $p_k(t)$ obtained in the induction process satisfies item (iii).

Finally, to prove item (iv), it suffices to take the $(k+1)$-th derivative of (50) at $t = 0$ and consider once again Faà di Bruno's formula to obtain

$$\alpha'(0) g^{(k+1)}(0) = c \sum_{j=2}^{k+1} D^j f(0)\, B_{k+1,j}\left(\gamma_k'(0), \ldots, \gamma_k^{(k-j+2)}(0)\right) + c Df(0) \alpha^{(k+1)}(0)\;. \tag{78}$$

In parallel, since $a_1 \neq 0$, Lemma A.3 ensures that

$$f_{\gamma_k}^{(k+1)}(0) = \sum_{j=2}^{k+1} D^j f(0)\, B_{k+1,j}\left(\gamma_k'(0), \ldots, \gamma_k^{(k-j+2)}(0)\right) + Df(0)\gamma_k^{(k+1)}(0) \tag{79}$$

satisfies $w^T f_{\gamma_k}^{(k+1)}(0) \neq 0$. Multiplying (79) by c, subtracting the result from (78), and multiplying on the left by w^T, it follows that

$$(w^T \alpha'(0)) g^{(k+1)}(0) - w^T f_{\gamma_k}^{(k+1)}(0) = 0, \tag{80}$$

which ensures

$$g^{(k+1)}(0) = \frac{w^T f_{\gamma_k}^{(k+1)}(0)}{b\, w^T v} \neq 0\;. \tag{81}$$

□

Lemma 8.2. *Let $F(x, \mu)$ be a k-parameter family of functions, $c \in \mathbb{R}^*$, and define the corresponding family of maps $\Pi : (x, \mu) \mapsto x + cF(x, \mu)$. Suppose that:*

(i) F undergoes an underlying catastrophe of corank 1 and codimension k at $x = 0$ for $\mu = 0$;

(ii) $\Pi_0 : x \mapsto \Pi(x, 0)$ has one simple eigenvalue equal to 1 at $x = 0$, all others not lying on the unit circle.

Then if $(t, \mu) \mapsto (g(t, \mu), \mu)$ is a restriction of the extended map $\Pi_e : (x, \mu) \mapsto (\Pi(x, \mu), \mu)$ to its $(k+1)$-dimensional centre manifold at $(x, \mu) = (0, 0)$, in local coordinates for which $(t, \mu) = (0, 0)$ corresponds to $(x, \mu) = (0, 0)$, the following hold:

(i) $\frac{\partial g}{\partial t}(0, 0) = 1$;

(ii) $\frac{\partial^i g}{\partial t^i}(0, 0) = 0$ for all $i \in \{2, \ldots, k\}$;

(iii) $\frac{\partial^{k+1} g}{\partial t^{k+1}}(0, 0) \neq 0$;

(*iv*)

$$\mathcal{D}_g := \det \begin{bmatrix} \frac{\partial g}{\partial \mu_1}(0,0) & \frac{\partial g}{\partial \mu_2}(0,0) & \cdots & \frac{\partial g}{\partial \mu_k}(0,0) \\ \frac{\partial^2 g}{\partial \mu_1 \partial t}(0,0) & \frac{\partial^2 g}{\partial \mu_2 \partial t}(0,0) & \cdots & \frac{\partial^2 g}{\partial \mu_k \partial t}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k g}{\partial \mu_1 \partial t^{k-1}}(0,0) & \frac{\partial^k g}{\partial \mu_2 \partial t^{k-1}}(0,0) & \cdots & \frac{\partial^k g}{\partial \mu_k \partial t^{k-1}}(0,0) \end{bmatrix} \neq 0$$

Proof. Let the centre manifold $\mathcal{M}_e$ of Π_e at $(0,0)$ be locally given by $\{(\alpha(t,\mu),\mu) \in \mathbb{R}^n \times \mathbb{R}^k : t \in (-t_0,t_0),\ \mu \in U\}$, where $t_0 > 0$ and U is an open neighbourhood of $0 \in \mathbb{R}^k$. This can always be achieved, because $\mathcal{M}_e$ is tangential to the center subspace. Restricting F and Π to $\mu = 0$, we obtain f and Π_0 satisfying exactly the conditions set out in Lemma 8.1, from which items (*i*) to (*iii*) follow. We also obtain a polynomial $p_0(t) = a_1 t + \ldots + a_k \frac{t^k}{k!}$ such that the curve $\gamma(t) = \phi^{-1}(p_0(t),0)$ has the same k-jet as $t \mapsto \alpha(t,0)$, where ϕ is any diffeomorphism in the class $[\phi]$ for which $[f] = [M] \cdot [s^n_{1k}] \circ [\phi]$.

Observe that, if we set $x = (x_1,x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the hypothesis that F undergoes an underlying catastrophe of corank 1 and codimension k actually ensures the existence of diffeomorphism of parameters $\eta = (\eta_1,\ldots,\eta_k)$, and smooth families $Q_\mu(x)$ and $\psi_\mu(x_1,x_2) = ((\psi_\mu)_1(x_1,x_2),(\psi_\mu)_2(x_1,x_2))$ such that $Q_0(x) = M(x)$, $\psi_0(x_1,x_2) = \phi(x_1,x_2)$, and

$$F(x,\mu) = Q(x,\mu) \cdot \begin{bmatrix} [(\psi_\mu)_1(x_1,x_2)]^{k+1} + \eta_1(\mu)\,[(\psi_\mu)_1(x_1,x_2)]^{k-1} + \cdots + \eta_k(\mu) \\ (\psi_\mu)_2(x_1,x_2) \end{bmatrix} . \tag{82}$$

Observe that, for small μ the function $(x_1,x_2) \mapsto \psi_\mu(x_1,x_2)$ is invertible, because ϕ is. The inverse of this function will be denoted by ψ_μ^{-1}.

The definition of g and the invariance of the centre manifold yield

$$\alpha(g(t,\mu),\mu) = \Pi(\alpha(t,\mu),\mu) = \alpha(t,\mu) + cF(\alpha(t,\mu),\mu) , \tag{83}$$

or, equivalently,

$$\alpha(g(t,\mu),\mu) - \alpha(t,\mu) = cF(\alpha(t,\mu),\mu) . \tag{84}$$

We let v and w^T be, respectively, right and left eigenvectors associated to the zero eigenvalue of $Df(0) = \frac{\partial F}{\partial x}(0,0)$. In order to obtain the determinant appearing in item (*iv*), we will make use of an inductive process to successively approximate relevant jet sections of α. First, we let $i \in \{1,2,\ldots,k\}$ and take the derivative of (84) with respect to μ_i at $\mu = 0$, obtaining

$$\frac{\partial \alpha}{\partial t}(g(t,0),0)\frac{\partial g}{\partial \mu_i}(t,0) + \left(\frac{\partial \alpha}{\partial \mu_i}(g(t,0),0) - \frac{\partial \alpha}{\partial \mu_i}(t,0)\right) = c\frac{\partial}{\partial \mu_i}\left(F(\alpha(t,\mu),\mu)\right)\Bigg|_{\mu=0} . \tag{85}$$

In particular, evaluating at $t = 0$ and taking the product on the left by w^T, it follows that

$$\left(w^T \frac{\partial \alpha}{\partial t}(0,0)\right)\frac{\partial g}{\partial \mu_i}(0,0) = cw^T\frac{\partial F}{\partial \mu_i}(0,0) . \tag{86}$$

Since there is $b \neq 0$ such that

$$\frac{\partial \alpha}{\partial t}(0,0) = bv , \tag{87}$$

Lemma A.2 ensures that

$$\frac{\partial g}{\partial \mu_i}(0,0) = \frac{c}{b(w^T v)} w^T \frac{\partial F}{\partial \mu_i}(0,0) \ . \tag{88}$$

Now, we define the first approximation of α by

$$\xi_0(t,\mu) := \psi_\mu^{-1}(p_0(t),0) \ . \tag{89}$$

One can easily see that $\xi_0(t,0) = \gamma(t)$, whose k-jet at $t=0$ agrees with $\alpha(t,0)$. However, there is no reason for the derivatives of ξ_0 and α with respect to μ to agree at $(0,0)$. The initial step of our iterative process will rectify this, but first we will use ξ_0 to find an expression for $\frac{\partial g}{\partial \mu_i}(0,0)$. By the definition of ξ_0 combined with (82), we know that

$$F(\xi_0(t,\mu),\mu) = Q_\mu(\xi_0(t,\mu)) \begin{bmatrix} (p_0(t))^{k+1} + \eta_1(\mu)(p_0(t))^{k-1} + \cdots + \eta_k(\mu) \\ 0 \end{bmatrix} , \tag{90}$$

so that taking the derivative with respect to μ_i at $\mu = 0$, it follows that

$$\frac{\partial F}{\partial x}(0,0)\frac{\partial \xi_0}{\partial \mu_i}(0,0) + \frac{\partial F}{\partial \mu_i}(0,0) = \frac{\partial \eta_k}{\partial \mu_i}(0) M(0) e_1 \ , \tag{91}$$

which, when multiplied on the left by w^T and combined with (88), yields

$$\frac{\partial g}{\partial \mu_i}(0,0) = \nu_0^0 \frac{\partial \eta_k}{\partial \mu_i}(0) \ , \tag{92}$$

where

$$\nu_0^0 := \frac{c}{b(w^T v)} \left(w^T M(0) e_1 \right) \ , \tag{93}$$

which is non-zero by Lemma A.3. The fact that v_0^0 does not vanish will be important in the proof.

Now, we proceed to actually constructing ξ_1, the next approximation of our iterative process. We evaluate (85) at $t=0$, obtaining

$$\frac{\partial \alpha}{\partial t}(0,0)\frac{\partial g}{\partial \mu_i}(0,0) = c\frac{\partial F}{\partial x}(0,0)\frac{\partial \alpha}{\partial \mu_i}(0,0) + c\frac{\partial F}{\partial \mu_i}(0,0) \ , \tag{94}$$

which can be combined with (91) to yield

$$\frac{\partial F}{\partial x}(0,0)\left(\frac{\partial \alpha}{\partial \mu_i}(0,0) - \frac{\partial \xi_0}{\partial \mu_i}(0,0)\right) = \frac{1}{c}\frac{\partial g}{\partial \mu_i}(0,0)\frac{\partial \alpha}{\partial t}(0,0) - \frac{\partial \eta_k}{\partial \mu_i}(0) M(0) e_1 \ . \tag{95}$$

Hence, if E is the matrix appearing in Lemma A.2 associated to the matrix $\frac{\partial F}{\partial x}(0,0)$ having one simple eigenvalue equal to zero, it follows that there is $r_i^0 \in \mathbb{R}$ such that

$$\frac{\partial \alpha}{\partial \mu_i}(0,0) = \frac{\partial \xi_0}{\partial \mu_i}(0,0) - \frac{\partial \eta_k}{\partial \mu_i} E M(0) e_1 + r_i^0 v \ . \tag{96}$$

Before we properly define ξ_1, there is one observation that must be made. By (82), if we take $(x,\mu) = (\phi^{-1}(t,0),0)$, it follows that

$$f(\phi^{-1}(t,0)) = F(\phi^{-1}(t,0),0) = M(\phi^{-1}(t,0)) \begin{bmatrix} t^{k+1} \\ 0 \end{bmatrix} \ . \tag{97}$$

Hence, by taking the derivative at $t = 0$, we obtain

$$Df(0)D(\phi^{-1})(0)e_1 = 0 \ . \tag{98}$$

Since $\ker(Df(0)) = \langle v \rangle$, there is $d_\phi \in \mathbb{R}$ such that

$$D(\phi^{-1})(0)e_1 = d_\phi v \ . \tag{99}$$

With that in mind, define

$$\begin{aligned} A_0(\mu) &= \sum_{i=1}^{k} \left[\frac{r_i^0}{d_\phi} - \frac{\partial \eta_k}{\partial \mu_i}(0) \left({e_1}^T D\phi(0) E M(0) e_1 \right) \right] \mu_i \in \mathbb{R}, \\ B_0(\mu) &= \sum_{i=1}^{k} \left[- \frac{\partial \eta_k}{\partial \mu_i}(0) \left(L_{n-1} \, D\phi(0) E M(0) e_1 \right) \right] \mu_i \in \mathbb{R}^{n-1} \ , \end{aligned} \tag{100}$$

where L_{n-1} is the $(n-1) \times n$ matrix

$$\left[\begin{array}{c|c} 0 & \\ \vdots & I_{n-1} \\ 0 & \end{array} \right] . \tag{101}$$

Then define $p_1(t, \mu) = p_0(t) + A_0(\mu)$, $q_1(t, \mu) = B_0(\mu)$, and

$$\xi_1(t, \mu) = \psi_\mu^{-1}(p_1(t, \mu), q_1(t, \mu)) = \psi_\mu^{-1}(p_0(t) + A_0(\mu), B_0(\mu)) \ . \tag{102}$$

Observe that

$$\begin{aligned} \frac{\partial \xi_1}{\partial \mu_i}(0, 0) &= \frac{\partial \psi_\mu^{-1}}{\partial \mu_i}\Big|_{\mu=0}(0, 0) + D(\phi^{-1})(0) \left[\begin{array}{c} \frac{\partial A_0}{\partial \mu_i}(0) \\ \frac{\partial B_0}{\partial \mu_i}(0) \end{array} \right] \\ &= \frac{\partial \xi_0}{\partial \mu_i}(0, 0) + r_i^0 v - \frac{\partial \eta_k}{\partial \mu_i}(0) E M(0) e_1 \\ &= \frac{\partial \alpha}{\partial \mu_i}(0, 0) \ , \end{aligned} \tag{103}$$

by (96). We have thus achieved the second iteration in the approximation.

The construction of subsequent terms is done through an induction process, over which we prove that the elements of the matrix appearing in item (iv) have a form amenable to treatment. More precisely, we will prove that, for each $N \in \{0, 1, \ldots, k-1\}$, there are $\nu_j^N \in \mathbb{R}$, with $j \in \{0, 1, \ldots, N\}$, such that $\nu_N^N \neq 0$ and

$$\frac{\partial^{N+1} g}{\partial \mu_i \partial t^N}(0, 0) = \sum_{j=0}^{N} \nu_j^N \frac{\partial \eta_{k-j}}{\partial \mu_i}(0) \tag{104}$$

for all $i \in \{1, \ldots, k\}$.

As part of the induction process, we also prove that, for each $N \in \{0, 1, \ldots, k-1\}$, there are

$$\begin{aligned} p_{N+1}(t, \mu) &= p_0(t) + A_0(\mu) + A_1(\mu) t + \ldots + A_N(\mu) t^N, \\ q_{N+1}(t, \mu) &= B_0(\mu) + B_1(\mu) t + \ldots + B_N(\mu) t^N \ , \end{aligned} \tag{105}$$

such that

$$A_j(0) = 0, \qquad B_j(\mu) = \sum_{i=1}^{k} \left[\sum_{l=0}^{j} S_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \right] \mu_i \ , \tag{106}$$

and

$$\xi_{N+1}(t,\mu) := \psi_\mu^{-1}(p_{N+1}(t,\mu), q_{N+1}(t,\mu)) \tag{107}$$

satisfies

$$\frac{\partial^{j+1}\xi_{N+1}}{\partial \mu_i \partial t^j}(0,0) = \frac{\partial^{j+1}\alpha}{\partial \mu_i \partial t^j}(0,0) \tag{108}$$

for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{0, \ldots, N\}$. Note that $S_l^j \in \mathbb{R}^{n-1}$ are to be determined and that, since we have shown that $\nu_0^0 \neq 0$, the base case $N = 0$ was already verified above, with

$$S_0^0 = -L_{n-1} D\phi(0) E M(0) e_1 \ . \tag{109}$$

Accordingly, let $N_* \in \{1, \ldots, k\}$ be given and suppose that (104) holds for all $N \in \{0, 1, \ldots, N_* - 1\}$, and that $p_{N+1}(t,\mu)$, $q_{N+1}(t,\mu)$ and $\xi_{N+1}(t,\mu)$ exist as above for those values of N as well. We have

$$\begin{aligned} \frac{\partial^{N_*+1}}{\partial \mu_i \partial t^{N_*}} \left(F(\alpha(t,\mu),\mu) \right) \Bigg|_{(t,\mu)=(0,0)} &= \frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial x}(\alpha(t,0),0) \frac{\partial \alpha}{\partial \mu_i}(t,0) + \frac{\partial F}{\partial \mu_i}(\alpha(t,0),0) \right) \Big|_{t=0} \\ &= \sum_{j=0}^{N_*} \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}} \left(\frac{\partial F}{\partial x}(\alpha(t,0),0) \right) \Big|_{t=0} \left(\frac{\partial^{j+1}\alpha}{\partial \mu_i \partial t^j}(0,0) \right) \\ &\quad + \frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\alpha(t,0),0) \right) \Big|_{t=0} \end{aligned} \tag{110}$$

Now, taking into account that $\xi_{N_*}(t,0) = \phi^{-1}(p_0(t),0) = \gamma(t)$, whose k-jet at $t = 0$ is identical to the k-jet of $t \mapsto \alpha(t,0)$ at $t = 0$, it follows that

$$\frac{d^{N_*-j}}{dt^{N_*-j}} \left(\frac{\partial F}{\partial x}(\alpha(t,0),0) \right) \Big|_{t=0} = \frac{d^{N_*-j}}{dt^{N_*-j}} \left(\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} \tag{111}$$

for all $j \in \{0, 1, \ldots, N_*\}$. Similarly,

$$\frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\alpha(t,0),0) \right) \Big|_{t=0} = \frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} . \tag{112}$$

Therefore, considering also (108), we obtain

$$\begin{aligned} &\frac{\partial^{N_*+1}}{\partial \mu_i \partial t^{N_*}} \left(F(\alpha(t,\mu),\mu) \right) \Bigg|_{(t,\mu)=(0,0)} \\ &\quad = \frac{\partial F}{\partial x}(0,0) \frac{\partial^{N_*+1}\alpha}{\partial \mu_i \partial t^{N_*}}(0,0) \\ &\qquad + \sum_{j=0}^{N_*-1} \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}} \left(\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} \left(\frac{\partial^{j+1}\xi_{N_*}}{\partial \mu_i \partial t^j}(0,0) \right) \\ &\qquad + \frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} . \end{aligned} \tag{113}$$

Now, using Leibniz's rule to take the N_*-th derivative of (85) at $t = 0$, it follows that

$$c\frac{\partial^{N_*+1}}{\partial\mu_i\partial t^{N_*}}\left(F(\alpha(t,\mu),\mu)\right)\Big|_{(t,\mu)=(0,0)} = \sum_{j=0}^{N_*}\binom{N_*}{j}\frac{\partial^{N*-j+1}g}{\partial\mu_i\partial t^{N_*-j}}(0,0)\,\frac{d^j}{dt^j}\left(\frac{\partial\alpha}{\partial t}(g(t,0),0)\right)\Big|_{t=0} + \frac{d^{N_*}}{dt^{N_*}}\left(\frac{\partial\alpha}{\partial\mu_i}(g(t,0),0)\right)\Big|_{t=0} - \frac{\partial^{N_*+1}\alpha}{\partial\mu_i\partial t^{N_*}}(0,0)\;. \tag{114}$$

Taking into account the already proved items (i) and (ii), we have

$$\frac{d^j}{dt^j}\left(\frac{\partial\alpha}{\partial t}(g(t,0),0)\right)\Big|_{t=0} = \frac{\partial^{j+1}\alpha}{\partial t^{j+1}}(0,0) \tag{115}$$

for $j \in \{0,\ldots,k\}$, and

$$\frac{d^{N_*}}{dt^{N_*}}\left(\frac{\partial\alpha}{\partial\mu_i}(g(t,0),0)\right)\Big|_{t=0} = \frac{\partial^{N_*+1}\alpha}{\partial\mu_i\partial t^{N_*}}(0,0)\;. \tag{116}$$

Hence, it follows that

$$c\frac{\partial^{N_*+1}}{\partial\mu_i\partial t^{N_*}}\left(F(\alpha(t,\mu),\mu)\right)\Big|_{(t,\mu)=(0,0)} = \sum_{j=0}^{N_*}\binom{N_*}{j}\frac{\partial^{N_*-j+1}g}{\partial\mu_i\partial t^{N_*-j}}(0,0)\,\left(\frac{\partial^{j+1}\alpha}{\partial t^{j+1}}(0,0)\right)\;. \tag{117}$$

Combined with (113), we obtain

$$\begin{aligned}\sum_{j=0}^{N_*}\frac{1}{c}\binom{N_*}{j}&\frac{\partial^{N_*-j+1}g}{\partial\mu_i\partial t^{N_*-j}}(0,0)\,\left(\frac{\partial^{j+1}\alpha}{\partial t^{j+1}}(0,0)\right)\\ &= \frac{\partial F}{\partial x}(0,0)\frac{\partial^{N_*+1}\alpha}{\partial\mu_i\partial t^{N_*}}(0,0)\\ &\quad+\sum_{j=0}^{N_*-1}\binom{N_*}{j}\frac{d^{N_*-j}}{dt^{N_*-j}}\left(\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0)\right)\Big|_{t=0}\left(\frac{\partial^{j+1}\xi_{N_*}}{\partial\mu_i\partial t^j}(0,0)\right)\\ &\quad+\frac{d^{N_*}}{dt^{N_*}}\left(\frac{\partial F}{\partial\mu_i}(\xi_{N_*}(t,0),0)\right)\Big|_{t=0}\;.\end{aligned} \tag{118}$$

The definition of ξ_{N_*} combined with (82), allows us to conclude that

$$F(\xi_{N_*}(t,\mu),\mu) = Q_\mu(\xi_{N_*}(t,\mu))\begin{bmatrix}(p_{N_*}(t,\mu))^{k+1}+\cdots+\eta_k(\mu)\\ q_{N_*}(t,\mu)\end{bmatrix}\;. \tag{119}$$

Thus, taking the derivative of both sides with respect to μ_i at $\mu = 0$, it follows that

$$\begin{aligned}\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0)\frac{\partial\xi_{N_*}}{\partial\mu_i}(0,0)+\frac{\partial F}{\partial\mu_i}(\xi_{N_*}(t,0),0)\\ = M(\xi_{N_*}(t,0))\begin{bmatrix}\delta_i p_{N_*}(t)\\ \delta_i q_{N_*}(t)\end{bmatrix}+R_{N_*}(t)e_1\;,\end{aligned} \tag{120}$$

where $\delta_i p_{N_*}$, $\delta_i q_{N_*}$, and $R_{N_*}(t)$ are defined by

$$\delta_i p_{N_*}(t) = (k+1)(p_0(t))^k \frac{\partial p_{N_*}}{\partial \mu_i}(t,0) + \sum_{l=0}^{k-1} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0)(p_0(t))^l, \tag{121}$$

$$\delta_i q_{N_*}(t) = \frac{\partial q_{N_*}}{\partial \mu_i}(t,0) = \frac{\partial B_0}{\partial \mu_i}(0) + \frac{\partial B_1}{\partial \mu_i}(0)t + \ldots + \frac{\partial B_{N_*-1}}{\partial \mu_i}(0)t^{N_*-1}, \tag{122}$$

$$R_{N_*}(t) = \frac{\partial Q}{\partial \mu}\Big|_{t=0}(\xi_{N_*}(t,0))(p_0(t))^{k+1}\ . \tag{123}$$

In particular, $R_{N_*}^{(j)}(0) = 0$ and the following identities hold for any $j \in \{0, 1, \ldots, N_*\}$:

$$\left(\delta_i p_{N_*}\right)^{(j)}(0) = \sum_{l=0}^{j} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \frac{d^j}{dt^j}\left((p_0(t))^l\right)\Big|_{t=0} = \sum_{l=0}^{j} C_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0), \tag{124}$$

$$\left(\delta_i q_{N_*}\right)^{(j)}(0) = \frac{\partial^{j+1} q_{N_*}}{\partial \mu_i \partial t^j}(0,0) = \begin{cases} j!\frac{\partial B_j}{\partial \mu_i}(0), \text{ if } j < N_*, \\ 0, \text{ if } j = N_*, \end{cases} \tag{125}$$

where C_l^j has been defined as

$$C_l^j := \frac{d^j}{dt^j}\left((p_0(t))^l\right)\Big|_{t=0}\ . \tag{126}$$

Making use of (106), we can rewrite the expression for $(\delta_i q_{N_*})^{(j)}$, for $j \in \{0, 1, \ldots, N_*\}$ as

$$(\delta_i q_{N_*})^{(j)}(0) = j!\frac{\partial B_j}{\partial \mu_i}(0)(1-\delta_{jN_*}) = \sum_{l=0}^{j} j!\, S_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0)(1-\delta_{jN_*})\ , \tag{127}$$

where δ_{jN_*} is the Kronecker delta.

Now, we take the N_*-th derivative of (120) at $t=0$, obtaining

$$\begin{aligned} \sum_{j=0}^{N_*} \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}}\left(M(\xi_{N_*}(t,0))\right)\Big|_{t=0} \begin{bmatrix} \left(\delta_i p_{N_*}\right)^{(j)}(0) \\ \left(\delta_i q_{N_*}\right)^{(j)}(0) \end{bmatrix} \\ = \frac{\partial F}{\partial x}(0,0)\frac{\partial^{N_*+1}\xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) \\ + \sum_{j=0}^{N_*-1} \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}}\left(\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0)\right)\Big|_{t=0}\left(\frac{\partial^{j+1}\xi_{N_*}}{\partial \mu_i \partial t^j}(0,0)\right) \\ + \frac{d^{N_*}}{dt^{N_*}}\left(\frac{\partial F}{\partial \mu_i}(\xi_{N_*}(t,0),0)\right)\Big|_{t=0}\ . \end{aligned} \tag{128}$$

Taking into account (124), (125) and (127), it follows that

$$\begin{bmatrix} \left(\delta_i p_{N_*}\right)^{(j)}(0) \\ \left(\delta_i q_{N_*}\right)^{(j)}(0) \end{bmatrix} = \begin{bmatrix} \sum_{l=0}^{j} C_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \\ \sum_{l=0}^{j} j! S_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0)(1-\delta_{jN_*}) \end{bmatrix} = \sum_{l=0}^{j} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \begin{bmatrix} C_l^j \\ j! S_l^j (1-\delta_{jN_*}) \end{bmatrix}\ , \tag{129}$$

so that, defining

$$Z_{l,j}^{N_*} := \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}} \left(M(\xi_{N_*}(t,0))\right)\big|_{t=0} \left[\begin{array}{c} C_l^j \\ j! S_l^j (1-\delta_{jN_*}) \end{array} \right] , \tag{130}$$

the left-hand side of (128) becomes

$$\sum_{j=0}^{N_*} \sum_{l=0}^{j} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_{l,j}^{N_*} = \sum_{l=0}^{N_*} \sum_{j=l}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_{l,j}^{N_*} = \sum_{l=0}^{N^*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_l^{N_*} , \tag{131}$$

where

$$Z_l^{N_*} := \sum_{j=l}^{N_*} Z_{l,j}^{N_*} . \tag{132}$$

Hence, we can rewrite (128) as

$$\begin{aligned} \sum_{l=0}^{N^*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_l^{N_*} &= \frac{\partial F}{\partial x}(0,0) \frac{\partial^{N_*+1} \xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) \\ &\quad + \sum_{j=0}^{N_*-1} \binom{N_*}{j} \frac{d^{N_*-j}}{dt^{N_*-j}} \left(\frac{\partial F}{\partial x}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} \left(\frac{\partial^{j+1} \xi_{N_*}}{\partial \mu_i \partial t^j}(0,0) \right) \\ &\quad + \frac{d^{N_*}}{dt^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\xi_{N_*}(t,0),0) \right) \Big|_{t=0} . \end{aligned} \tag{133}$$

By subtracting (133) from (118), it follows that

$$\begin{aligned} &\frac{\partial F}{\partial x}(0,0) \left(\frac{\partial^{N_*+1} \alpha}{\partial \mu_i \partial t^{N_*}}(0,0) - \frac{\partial^{N_*+1} \xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) \right) = \\ &\sum_{j=0}^{N_*} \frac{1}{c} \binom{N_*}{j} \frac{\partial^{N_*-j+1} g}{\partial \mu_i \partial t^{N_*-j}}(0,0) \left(\frac{\partial^{j+1} \alpha}{\partial t^{j+1}}(0,0) \right) - \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_l^{N_*} . \end{aligned} \tag{134}$$

Then defining

$$d_t \alpha_{N_*-j}^{N_*} := \binom{N_*}{j} \left(\frac{\partial^{j+1} \alpha}{\partial t^{j+1}}(0,0) \right) , \tag{135}$$

and taking $j \mapsto N_* - j$, the first sum on the right-hand side can be more conveniently written as

$$\frac{1}{c} \frac{\partial \alpha}{\partial t}(0,0) \frac{\partial^{N_*+1} g}{\partial \mu_i \partial t^{N_*}}(0,0) + \sum_{j=0}^{N_*-1} \frac{1}{c} \frac{\partial^{j+1} g}{\partial \mu_i \partial t^j}(0,0)\, d_t \alpha_j^{N_*} . \tag{136}$$

By induction hypothesis, each of the derivatives of g appearing in the sum can be rewritten as in (104), so that the sum appearing above becomes

$$\sum_{j=0}^{N_*-1} \sum_{l=0}^{j} \frac{\nu_l^j}{c} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) d_t \alpha_j^{N_*} = \sum_{l=0}^{N_*-1} \sum_{j=l}^{N_*-1} \frac{\nu_l^j}{c} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0)\, d_t \alpha_j^{N_*} . \tag{137}$$

Thus, we define

$$Y_l^{N_*} := \sum_{j=l}^{N_*-1} \nu_l^j \, d_t \alpha_j^{N_*} \ , \tag{138}$$

for $l \in \{0, 1, \ldots, N_* - 1\}$ and $Y_{N_*}^{N_*} = 0$, and the first sum on the right-hand side of (134) can be rewritten as

$$\frac{1}{c}\frac{\partial \alpha}{\partial t}(0,0)\frac{\partial^{N_*+1} g}{\partial \mu_i \partial t^{N_*}}(0,0) + \sum_{l=0}^{N_*-1} \frac{1}{c}\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Y_l^{N_*} \ , \tag{139}$$

so that (134) becomes

$$\begin{aligned} &\frac{\partial F}{\partial x}(0,0)\left(\frac{\partial^{N_*+1}\alpha}{\partial \mu_i \partial t^{N_*}}(0,0) - \frac{\partial^{N_*+1}\xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0)\right) = \\ &\qquad \frac{1}{c}\frac{\partial \alpha}{\partial t}(0,0)\frac{\partial^{N_*+1} g}{\partial \mu_i \partial t^{N_*}}(0,0) + \sum_{l=0}^{N_*-1} \frac{1}{c}\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Y_l^{N_*} - \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_l^{N_*} \ . \end{aligned} \tag{140}$$

The first statement of the induction process can now be easily proved. Multiplying (140) on the left by w^T and rearranging, we obtain

$$\frac{\partial^{N_*+1} g}{\partial \mu_i \partial^{N_*}}(0,0) = \sum_{l=0}^{N_*} \frac{1}{b(w^T v)}\left(c w^T Z_l^{N_*} - w^T Y_l^{N_*}\right)\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \ , \tag{141}$$

and the first part of the induction follows by defining

$$\nu_l^{N_*} := \frac{1}{b(w^T v)}\left(c w^T Z_l^{N_*} - w^T Y_l^{N_*}\right) \ , \tag{142}$$

for $l \in \{0, \ldots, N_*\}$. We still have to verify that $\nu_{N_*}^{N_*} \neq 0$, which holds because, considering the definitions of Z_l^j and C_l^j, and also that $\nu_0^0 \neq 0$ and $p_0'(0) \neq 0$, we have

$$\nu_{N_*}^{N_*} = \frac{1}{b(w^T v)} c w^T Z_{N_*}^{N_*} = \frac{c}{b(w^T v)} w^T M(0) \begin{bmatrix} C_{N_*}^{N_*} \\ 0 \end{bmatrix} = (N_*)!(p_0'(0))^{N_*}\nu_0^0 \neq 0 \tag{143}$$

The second part of the induction, which consists in constructing p_{N_*+1}, q_{N_*+1}, and ξ_{N_*+1} is as follows. By (140) and Lemma A.2, we conclude that there is $r_i^{N_*} \in \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial^{N_*+1}\alpha}{\partial \mu_i \partial t^{N_*}}(0,0) &= \frac{\partial^{N_*+1}\xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) + r_i^{N_*} v + \sum_{l=0}^{N_*-1} \frac{1}{c}\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) E Y_l^{N_*} - \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) E Z_l^{N_*} \\ &= \frac{\partial^{N_*+1}\xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) + r_i^{N_*} v + \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) E\left(\frac{1}{c} Y_l^{N_*} - Z_l^{N_*}\right) \ . \end{aligned} \tag{144}$$

With that in mind, we define

$$\begin{aligned} A_{N_*}(\mu) &= \frac{1}{(N_*)!}\sum_{i=1}^{k}\left[\frac{r_i^{N_*}}{d_\phi} + \sum_{l=0}^{N_*}\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) {e_1}^T D\phi(0) E\left(\frac{1}{c} Y_l^{N_*} - Z_l^{N_*}\right)\right]\mu_i \in \mathbb{R}, \\ B_{N_*}(\mu) &= \frac{1}{(N_*)!}\sum_{i=1}^{k}\left[\sum_{l=0}^{N_*}\frac{\partial \eta_{k-l}}{\partial \mu_i}(0) L_{n-1} D\phi(0) E\left(\frac{1}{c} Y_l^{N_*} - Z_l^{N_*}\right)\right]\mu_i \in \mathbb{R}^{n-1} \ , \end{aligned} \tag{145}$$

and

$$\begin{aligned}
p_{N_*+1}(t,\mu) &= p_{N_*}(t,\mu) + A_{N_*}(\mu)t^{N_*} = p_0(t) + A_0(\mu) + \ldots + A_{N_*}(\mu)t^{N_*}, \\
q_{N_*+1}(t,\mu) &= q_{N_*}(t,\mu) + B_{N_*}(\mu)t^{N_*} = B_0(\mu) + \ldots + B_{N_*}(\mu)t^{N_*}, \\
\xi_{N_*+1}(t,\mu) &= \psi_\mu^{-1}(p_{N_*+1}(t,\mu), q_{N_*+1}(t,\mu)) \ .
\end{aligned} \tag{146}$$

By definition of ξ_{N_*+1}, we obtain

$$\frac{\partial \xi_{N_*+1}}{\partial \mu_i}(t,0) = \frac{\partial \psi_\mu^{-1}}{\partial \mu_i}\Big|_{\mu=0}(p_0(t),0) + D(\phi^{-1})(0) \begin{bmatrix} \frac{\partial A_0}{\partial \mu_i}(0) + \ldots + \frac{\partial A_{N_*}}{\partial \mu_i}(0)t^{N_*} \\ \frac{\partial B_0}{\partial \mu_i}(0) + \ldots + \frac{\partial B_{N_*}}{\partial \mu_i}(0)t^{N_*} \end{bmatrix} . \tag{147}$$

Similarly, we have

$$\frac{\partial \xi_{N_*}}{\partial \mu_i}(t,0) = \frac{\partial \psi_\mu^{-1}}{\partial \mu_i}\Big|_{\mu=0}(p_0(t),0) + D(\phi^{-1})(0) \begin{bmatrix} \frac{\partial A_0}{\partial \mu_i}(0) + \ldots + \frac{\partial A_{N_*-1}}{\partial \mu_i}(0)t^{N_*-1} \\ \frac{\partial B_0}{\partial \mu_i}(0) + \ldots + \frac{\partial B_{N_*-1}}{\partial \mu_i}(0)t^{N_*-1} \end{bmatrix} , \tag{148}$$

so that

$$\frac{\partial \xi_{N_*+1}}{\partial \mu_i}(t,0) - \frac{\partial \xi_{N_*}}{\partial \mu_i}(t,0) = D(\phi^{-1})(0) \begin{bmatrix} \frac{\partial A_{N_*}}{\partial \mu_i}(0)t^{N_*} \\ \frac{\partial B_{N_*}}{\partial \mu_i}(0)t^{N_*} \end{bmatrix} . \tag{149}$$

It is thus clear that the (N_*-1)-jets of $t \mapsto \frac{\partial \xi_{N_*+1}}{\partial \mu_i}(t,0)$ and $t \mapsto \frac{\partial \xi_{N_*}}{\partial \mu_i}(t,0)$ are equal. Since, by induction hypothesis, the (N_*-1)-jet of $t \mapsto \frac{\partial \xi_{N_*}}{\partial \mu_i}(t,0)$ agrees with that of $t \mapsto \frac{\partial \alpha}{\partial \mu_i}(t,0)$, it follows that

$$\frac{\partial^{j+1} \xi_{N_*+1}}{\partial \mu_i \partial t^j}(0,0) = \frac{\partial^{j+1} \alpha}{\partial \mu_i \partial t^j}(0,0) \tag{150}$$

for all $i \in \{1,2,\ldots,k\}$ and all $j \in \{0,\ldots,N_*-1\}$. Finally, by taking the N_*-th derivative of (149) at $t=0$ and considering the definitions of A_{N_*} and B_{N_*}, as well as (144), we also obtain

$$\begin{aligned}
\frac{\partial^{N_*+1} \xi_{N_*+1}}{\partial \mu_i \partial t^{N_*}}(0,0) &= \frac{\partial^{N_*+1} \xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) + D(\phi^{-1})(0) \begin{bmatrix} (N_*)! \, \frac{\partial A_{N_*}}{\partial \mu_i}(0) \\ (N_*)! \, \frac{\partial B_{N_*}}{\partial \mu_i}(0) \end{bmatrix} \\
&= \frac{\partial^{N_*+1} \xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0) + r_i^{N_*} v + \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) E\left(\frac{1}{c} Y_l^{N_*} - Z_l^{N_*}\right) \\
&= \frac{\partial^{N_*+1} \alpha}{\partial \mu_i \partial t^{N_*}}(0,0) \ ,
\end{aligned} \tag{151}$$

which proves that ξ_{N_*+1} is as wished and concludes the induction.

We can now proceed to actually proving item (iv) using the statement we have verified by induction. Applying (104), it follows that the determinant $\mathcal{D}_g$ appearing in item (iv) satisifies

$$\mathcal{D}_g = \det \begin{bmatrix} v_0^0 \frac{\partial \eta_k}{\partial \mu}(0) \\ v_0^1 \frac{\partial \eta_k}{\partial \mu}(0) + v_1^1 \frac{\partial \eta_{k-1}}{\partial \mu}(0) \\ \vdots \\ v_0^{k-1} \frac{\partial \eta_k}{\partial \mu}(0) + v_1^{k-1} \frac{\partial \eta_{k-1}}{\partial \mu}(0) + \cdots + v_{k-1}^{k-1} \frac{\partial \eta_1}{\partial \mu}(0) \end{bmatrix} , \tag{152}$$

where each $\frac{\partial \eta_{k-l}}{\partial \mu}(0)$ represents the row

$$\left[\frac{\partial \eta_{k-l}}{\partial \mu_1}(0) \; \frac{\partial \eta_{k-l}}{\partial \mu_2}(0) \; \cdots \; \frac{\partial \eta_{k-l}}{\partial \mu_k}(0) \right] . \tag{153}$$

Thus, since the determinant can be seen as a multilinear alternating function of the rows of the matrix to which it is applied, it follows that

$$|\mathcal{D}_g| = \Pi_{i=0}^{k-1} |v_i^i| \left| \det(\frac{\partial \eta}{\partial \mu}(0)) \right| , \tag{154}$$

which does not vanish because η is a local diffeomorphism at $\mu = 0$. The proof is thus finished. □

8.2 Proof of Theorem 2.1

As with maps, we present the proof as two different results that imply that any family (of vector fields) satisfying the hypotheses given in Theorem 2.1 is minimally topologically stable, from which point Theorem 7.1 can be applied. Without loss of generality, we assume $(x_*, \mu_*) = (0, 0)$.

Lemma 8.3. *Let $[f]$ be the germ of an underlying catastrophe of corank 1 and codimension k at $x = 0$, i.e. there are $[M] \in GL_n(\mathcal{E}_n)$ and $[\phi] \in L_n$ such that $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$. Suppose that the derivative $Df(0)$ has exactly one simple eigenvalue equal to 0, all others having non-zero real parts. Define $\dot{y} = g(y)$ to be the restriction of $\dot{x} = f(x)$ to its 1-dimensional centre manifold at $x = 0$ in local coordinates such that $y = 0$ corresponds to $x = 0$. Then the following hold:*

(i) $g^{(i)}(0) = 0$ *for all* $i \in \{1, 2, \ldots, k\}$;

(ii) if $y \mapsto \alpha(y)$ is a regular parametrization of a smooth 1-dimensional centre manifold of f at $x = 0$, there is a polynomial $p(t) = a_1 t + \ldots + a_k \frac{t^k}{k!}$ such that $a_1 \neq 0$ and the curve $\gamma(y) := \phi^{-1}(p(y), 0)$ has the same k-jet at 0 as α;

(iii) $g^{(k+1)}(0) \neq 0$.

Proof. The proof follows along the same lines of the proof of Lemma 8.1, but notice that the centre manifold is an entirely different object in the present case, since it is obtained by looking at f as a vector field, whereas before we were interested in the centre manifold of a diffeomorphism having f as its displacement function.

Since $Df(0)$ has only one simple eigenvalue equal to zero with vanishing real part, it follows that the centre manifold $\mathcal{M}$ of $\dot{x} = f(x)$ at $x = 0$ is 1-dimensional and can be written in the form $\{\alpha(y) \in \mathbb{R}^n : y \in (-y_0, y_0)\}$, where $y_0 > 0$, $\alpha(0) = 0$, and $\alpha'(0) \neq 0$. Thus, a right eigenvector v of $Df(0)$ associated to 0 has to be a multiple of the derivative of α at 0, i.e., there is $b \neq 0$ such that

$$\alpha'(0) = b\, v . \tag{155}$$

We also define w^T to be a corresponding left eigenvector associated to zero.

Let $\dot{y} = g(y)$ denote the restriction of $\dot{x} = f(x)$ to $\mathcal{M}$ in local coordinates. Then it is clear that

$$\alpha'(y) g(y) = f(\alpha(y)) \tag{156}$$

Differentiating once with respect to t at 0, we obtain

$$\alpha'(0)g'(0) = Df(0)\alpha'(0) = b\, Df(0)v = 0 \tag{157}$$

because v is an eigenvector of $Df(0)$ associated to 0. Hence, it follows that

$$g'(0) = 0\ . \tag{158}$$

We will by induction that two different statements hold for $N \in \{1, 2, \ldots, k\}$:

(a) $g^{(N)}(0) = 0$;

(b) there is a polynomial

$$p_N(t) = a_1 t + a_2 \frac{t^2}{2} + \ldots + a_N \frac{t^N}{N!}$$

with $a_1 \neq 0$ such that the curve $\gamma_N(y) := \phi^{-1}(p_N(y), 0)$ satisfies $\gamma_N^{(j)}(0) = \alpha^{(j)}(0)$ for all $j \in \{0, 1, \ldots, N\}$.

We start with the base case $i = 1$, for which the first statement has been proved. By hypothesis, $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$, and we assume that representatives M of $[M]$ and ϕ of $[\phi]$ are chosen. Define $\gamma_0(y) := \phi^{-1}(y, 0)$. By Lemma A.3, it follows that

$$\frac{\partial f(\gamma_0(y))}{\partial y}\Big|_{y=0} = Df(0)\,\gamma_0'(0) = 0\ . \tag{159}$$

Thus, it follows that $\gamma_0'(0)$ is an eigenvector of $Df(0)$ associated to the eigenvalue 0. Since this eigenvalue is simple and $\alpha'(0)$ is also an eigenvector associated to it, there is $a_1 \in \mathbb{R}^*$ such that

$$\alpha'(0) = a_1 \gamma_0'(0)\ . \tag{160}$$

Hence, we define

$$\gamma_1(y) := \gamma_0\left(a_1 y\right) = \phi^{-1}\left(a_1 y, 0\right)\ , \tag{161}$$

and, considering (160), it follows that $\gamma_1'(0) = a_1\gamma_0'(0) = \alpha'(0)$. We have thus verified the base case of the induction.

Let a natural number N with $1 < N \leq k$ be given. Assume that $g^{(j)}(0) = 0$ and that

$$\gamma_{N-1}(y) = \phi^{-1}\left(a_1 y + a_2 \frac{y^2}{2} + \cdots + a_{N-1}\frac{y^{N-1}}{(N-1)!}, 0\right) \tag{162}$$

satisfies $\gamma_{N-1}^{(j)}(0) = \alpha^{(j)}(0)$ for all $j \in \{0, 1, \ldots, N-1\}$.

We take the N-th derivative of (156) with respect to y at $(0, 0)$:

$$\alpha'(0)g^{(N)}(0) = \frac{\partial^N}{\partial y^N} f(\alpha(y))\Big|_{y=0}\ . \tag{163}$$

This coincides with the identity (68) obtained for maps, except that now $c = 1$. We proceed exactly as we did in that case, making use of Faà di Bruno's formula to conclude that

$$\alpha^{(N)}(0) - \gamma_{N-1}^{(N)}(0) \in \ker\left(Df(0)\right) = \langle v \rangle = \langle \gamma_0'(0) \rangle\ , \tag{164}$$

so that there is $a_N \in \mathbb{R}$ such that

$$\alpha^{(N)}(0) = \gamma_{N-1}^{(N)}(0) + a_N \gamma_0'(0) = \gamma_{N-1}^{(N)}(0) + a_N \frac{\partial \phi^{-1}}{\partial x_1}(0,0) \ . \tag{165}$$

We then define

$$\gamma_N(y) := \phi^{-1}\left(a_1 y + \ldots + a_N \frac{y^N}{N!},\ 0\right) \tag{166}$$

and proceed as in the proof of Lemma 8.1 to conclude that

$$\gamma_N^{(N)}(0) = \gamma_{N-1}^{(N)}(0) + \frac{\partial \phi^{-1}}{\partial x_1}(0,0)\, a_N = \alpha^{(N)}(0) \ , \tag{167}$$

which terminates the induction argument and proves item (i). Once more, the polynomial

$$p_k(t) = a_1 t + a_2 \frac{y^2}{2} + \cdots + a_k \frac{t^k}{k!} \tag{168}$$

obtained in the induction process satisfies item (ii).

To prove item (iii), we take the $(k+1)$-th derivative of (50) at $y = 0$, as we did in Lemma 8.1, and consider Faà di Bruno's formula in conjuction with Lemma A.3 to obtain

$$g^{(k+1)}(0) = \frac{w^T f_{\gamma_k}^{(k+1)}(0)}{b\, w^T v} \neq 0 \ . \tag{169}$$

□

Lemma 8.4. *Let $F(x, \mu)$ be a k-parameter family of vector fields. Suppose that*

(i) F undergoes an underlying catastrophe of corank 1 and codimension k at $x = 0$ for $\mu = 0$;

(ii) $DF(0, 0)$ has one simple eigenvalue equal to 0, all others having non-zero real parts.

Then if

$$\dot{y} = g(y, \mu), \quad \dot{\mu} = 0 \tag{170}$$

is a restriction of the extended system

$$\dot{x} = F(x, \mu), \quad \dot{\mu} = 0 \tag{171}$$

to its $(k+1)$-dimensional centre manifold at $(x, \mu) = (0, 0)$ in local coordinates for which $(y, \mu) = (0, 0)$ corresponds to $(x, \mu) = (0, 0)$, the following hold:

(i) $\frac{\partial^i g}{\partial y^i}(0, 0) = 0$ for all $i \in \{1, \ldots, k\}$;

(ii) $\frac{\partial^{k+1} g}{\partial y^{k+1}}(0, 0) \neq 0$;

(*iii*)

$$\mathcal{D}_g := \det \begin{bmatrix} \frac{\partial g}{\partial \mu_1}(0,0) & \frac{\partial g}{\partial \mu_2}(0,0) & \cdots & \frac{\partial g}{\partial \mu_k}(0,0) \\ \frac{\partial^2 g}{\partial \mu_1 \partial y}(0,0) & \frac{\partial^2 g}{\partial \mu_2 \partial y}(0,0) & \cdots & \frac{\partial^2 g}{\partial \mu_k \partial y}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^k g}{\partial \mu_1 \partial y^{k-1}}(0,0) & \frac{\partial^k g}{\partial \mu_2 \partial y^{k-1}}(0,0) & \cdots & \frac{\partial^k g}{\partial \mu_k \partial y^{k-1}}(0,0) \end{bmatrix} \neq 0$$

Proof. Let the centre manifold $\mathcal{M}_e$ of (171) at $(0,0)$ be locally given by $\{(\alpha(y,\mu),\mu) \in \mathbb{R}^n \times \mathbb{R}^k : y \in (-y_0, y_0),\ \mu \in U\}$, where $y_0 > 0$ and U is an open neighbourhood of $0 \in \mathbb{R}^k$. This can always be done, because $\mathcal{M}_e$ is tangential to the center subspace of (171). Restricting F to $\mu = 0$, we obtain $\dot{x} = f(x) = F(x,0)$ satisfying exactly the conditions set out in Lemma 8.3, from which items (*i*) and (*ii*) follow. We also obtain a polynomial $p_0(t) = a_1 t + \ldots + a_k \frac{t^k}{k!}$ such that the curve $\gamma(y) = \phi^{-1}(p_0(y),0)$ has the same k-jet as $y \mapsto \alpha(y,0)$, where ϕ is any diffeomorphism in the class $[\phi]$ for which $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$.

As done in the map case, we set $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and the hypothesis that F undergoes an underlying catastrophe of corank 1 and codimension k ensures the existence of smooth families $Q_\mu(x)$ and $\psi_\mu(x_1,x_2) = ((\psi_\mu)_1(x_1,x_2), (\psi_\mu)_2(x_1,x_2))$ such that $Q_0(x) = M(x)$, $\psi_0(x_1,x_2) = \phi(x_1,x_2)$, and

$$F(x,\mu) = Q(x,\mu) \cdot \begin{bmatrix} [(\psi_\mu)_1(x_1,x_2)]^{k+1} + \eta_1(\mu)\,[(\psi_\mu)_1(x_1,x_2)]^{k-1} + \cdots + \eta_k(\mu) \\ (\psi_\mu)_2(x_1,x_2) \end{bmatrix} . \tag{172}$$

For small μ, the function $(x_1,x_2) \mapsto \psi_\mu(x_1,x_2)$ is invertible, and its inverse will be denoted by ψ_μ^{-1}.

The definition of g and the invariance of the centre manifold yield

$$\frac{\partial \alpha}{\partial y}(y,\mu)\, g(y,\mu) = F(\alpha(y,\mu),\mu) . \tag{173}$$

Let v and w^T be, respectively, right and left eigenvectors associated to the zero eigenvalue of $Df(0) = \frac{\partial F}{\partial x}(0,0)$. In order to obtain the determinant appearing in item (*iii*), we will make use of an inductive process akin to the one applied in the map case.

First, we let $i \in \{1,2,\ldots,k\}$ and take the derivative of (173) with respect to μ_i at $\mu = 0$, obtaining

$$\frac{\partial \alpha}{\partial y}(y,0)\frac{\partial g}{\partial \mu_i}(y,0) + \frac{\partial^2 \alpha}{\partial \mu_i \partial y}(y,0) g(y,0) = \frac{\partial}{\partial \mu_i}\left(F(\alpha(y,\mu),\mu)\right)\Bigg|_{\mu=0} . \tag{174}$$

In particular, evaluating at $y = 0$ and taking the product on the left by w^T, it follows that

$$\left(w^T \frac{\partial \alpha}{\partial y}(0,0)\right) \frac{\partial g}{\partial \mu_i}(0,0) = w^T \frac{\partial F}{\partial \mu_i}(0,0) . \tag{175}$$

Since there is $b \neq 0$ such that

$$\frac{\partial \alpha}{\partial t}(0,0) = bv . \tag{176}$$

Lemma A.2 ensures that

$$\frac{\partial g}{\partial \mu_i}(0,0) = \frac{1}{b(w^T v)} w^T \frac{\partial F}{\partial \mu_i}(0,0) \ , \tag{177}$$

exactly as in the map case, but with $c = 1$.

Now, we define the first approximation of α by

$$\xi_0(y,\mu) := \psi_\mu^{-1}(p_0(y), 0) \ . \tag{178}$$

One can see that $\xi_0(y,0) = \gamma(y)$, whose k-jet at $y = 0$ agrees with $\alpha(y,0)$. We can then use ξ_0 to find an expression for $\frac{\partial g}{\partial \mu_i}(0,0)$. We make use of the definition of ξ_0 combined with (172), as in the map case, to conclude that

$$\frac{\partial g}{\partial \mu_i}(0,0) = \nu_0^0 \frac{\partial \eta_k}{\partial \mu_i}(0) \ , \tag{179}$$

where

$$\nu_0^0 := \frac{1}{b(w^T v)} \left(w^T M(0) e_1 \right) \ , \tag{180}$$

which is non-zero by Lemma A.3.

As in the proof of Lemma 8.2, we proceed to constructing ξ_1, the next step of our iterative process. We evaluate (174) at $t = 0$, obtaining

$$\frac{\partial \alpha}{\partial y}(0,0) \frac{\partial g}{\partial \mu_i}(0,0) = \frac{\partial F}{\partial x}(0,0) \frac{\partial \alpha}{\partial \mu_i}(0,0) + \frac{\partial F}{\partial \mu_i}(0,0) \ , \tag{181}$$

which can be combined with (91), yielding

$$\frac{\partial F}{\partial x}(0,0) \left(\frac{\partial \alpha}{\partial \mu_i}(0,0) - \frac{\partial \xi_0}{\partial \mu_i}(0,0) \right) = \frac{\partial g}{\partial \mu_i}(0,0) \frac{\partial \alpha}{\partial y}(0,0) - \frac{\partial \eta_k}{\partial \mu_i}(0) M(0) e_1 \tag{182}$$

Since all identities are extremely similar to the map case, we conclude analogously that, if E is the matrix appearing in Lemma A.2, it follows that there is $r_i^0 \in \mathbb{R}$ such that

$$\frac{\partial \alpha}{\partial \mu_i}(0,0) = \frac{\partial \xi_0}{\partial \mu_i}(0,0) - \frac{\partial \eta_k}{\partial \mu_i} E M(0) e_1 + r_i^0 v \ . \tag{183}$$

Before we define ξ_1, we observe that, exactly as in the case of maps, there is $d_\phi \in \mathbb{R}$ such that

$$D(\phi^{-1})(0) e_1 = d_\phi v \ . \tag{184}$$

As before, define

$$\begin{aligned} A_0(\mu) &= \sum_{i=1}^{k} \left[\frac{r_i^0}{d_\phi} - \frac{\partial \eta_k}{\partial \mu_i}(0) \left({e_1}^T D\phi(0) E M(0) e_1 \right) \right] \mu_i \in \mathbb{R}, \\ B_0(\mu) &= \sum_{i=1}^{k} \left[-\frac{\partial \eta_k}{\partial \mu_i}(0) \left(L_{n-1} \, D\phi(0) E M(0) e_1 \right) \right] \mu_i \in \mathbb{R}^{n-1} \ , \end{aligned} \tag{185}$$

where L_{n-1} is the $(n-1) \times n$ matrix

$$\left[\begin{array}{c|c} 0 & \\ \vdots & I_{n-1} \\ 0 & \end{array} \right] . \tag{186}$$

Also, define $p_1(t,\mu) = p_0(t) + A_0(\mu)$, $q_1(t,\mu) = B_0(\mu)$, and

$$\xi_1(y,\mu) = \psi_\mu^{-1}(p_1(y,\mu), q_1(y,\mu)) = \psi_\mu^{-1}(p_0(y) + A_0(\mu), B_0(\mu)) , \tag{187}$$

so that

$$\begin{aligned} \frac{\partial \xi_1}{\partial \mu_i}(0,0) &= \frac{\partial \psi_\mu^{-1}}{\partial \mu_i}\Big|_{\mu=0}(0,0) + D(\phi^{-1})(0) \left[\begin{array}{c} \frac{\partial A_0}{\partial \mu_i}(0) \\ \frac{\partial B_0}{\partial \mu_i}(0) \end{array} \right] \\ &= \frac{\partial \xi_0}{\partial \mu_i}(0,0) + r_i^0 v - \frac{\partial \eta_k}{\partial \mu_i}(0) EM(0) e_1 \\ &= \frac{\partial \alpha}{\partial \mu_i}(0,0) , \end{aligned} \tag{188}$$

by (183). This is the second iteration of our process.

The construction of subsequent terms is, as with maps in Lemma 8.2, achieved via induction, in the course of which we prove that, for each $N \in \{0, 1, \ldots, k-1\}$, there are $\nu_j^N \in \mathbb{R}$, with $j \in \{0, 1, \ldots, N\}$, such that $\nu_N^N \neq 0$ and

$$\frac{\partial^{N+1} g}{\partial \mu_i \partial t^N}(0,0) = \sum_{j=0}^{N} \nu_j^N \frac{\partial \eta_{k-j}}{\partial \mu_i}(0) \tag{189}$$

for all $i \in \{1, \ldots, k\}$. The subsequent approximation terms themselves are obtained by showing that, for each $N \in \{0, 1, \ldots, k-1\}$, there are

$$\begin{aligned} p_{N+1}(t,\mu) &= p_0(t) + A_0(\mu) + A_1(\mu) t + \ldots + A_N(\mu) t^N, \\ q_{N+1}(t,\mu) &= B_0(\mu) + B_1(\mu) t + \ldots + B_N(\mu) t^N , \end{aligned} \tag{190}$$

such that

$$A_j(0) = 0, \qquad B_j(\mu) = \sum_{i=1}^{k} \left[\sum_{l=0}^{j} S_l^j \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) \right] \mu_i , \tag{191}$$

and

$$\xi_{N+1}(t,\mu) := \psi_\mu^{-1}(p_{N+1}(t,\mu), q_{N+1}(t,\mu)) \tag{192}$$

satisfies

$$\frac{\partial^{j+1} \xi_{N+1}}{\partial \mu_i \partial t^j}(0,0) = \frac{\partial^{j+1} \alpha}{\partial \mu_i \partial t^j}(0,0) \tag{193}$$

for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{0, \ldots, N\}$. As in the case of maps, $S_l^j \in \mathbb{R}^{n-1}$ are to be determined. The base case $N = 0$ is already done, with

$$S_0^0 = -L_{n-1} D\phi(0) EM(0) e_1 . \tag{194}$$

Accordingly, let $N_* \in \{1, \dots, k\}$ be given and suppose that (104) holds for all $N \in \{0, 1, \dots, N_*-1\}$, and that $p_{N+1}(t, \mu)$, $q_{N+1}(t, \mu)$ and $\xi_{N+1}(t, \mu)$ exist as above for those values of N as well.

We proceed as in the map case to conclude from (193) that

$$\begin{aligned} &\frac{\partial^{N_*+1}}{\partial \mu_i \partial y^{N_*}} \left(F(\alpha(y, \mu), \mu)\right)\Bigg|_{(y,\mu)=(0,0)} \\ &\quad = \frac{\partial F}{\partial x}(0,0) \frac{\partial^{N_*+1}\alpha}{\partial \mu_i \partial y^{N_*}}(0,0) \\ &\quad + \sum_{j=0}^{N_*-1} \binom{N_*}{j} \frac{d^{N_*-j}}{dy^{N_*-j}} \left(\frac{\partial F}{\partial x}(\xi_{N_*}(y,0),0)\right)\Big|_{y=0} \left(\frac{\partial^{j+1}\xi_{N_*}}{\partial \mu_i \partial y^j}(0,0)\right) \\ &\quad + \frac{d^{N_*}}{dy^{N_*}} \left(\frac{\partial F}{\partial \mu_i}(\xi_{N_*}(y,0),0)\right)\Big|_{y=0} . \end{aligned} \tag{195}$$

Also, an application of Leibniz's rule to take the N_*-th derivative of (174) at $y = 0$, in combination with the fact that item (i) has been proved already, yields

$$\frac{\partial^{N_*+1}}{\partial \mu_i \partial t^{N_*}} \left(F(\alpha(t, \mu), \mu)\right)\Bigg|_{(y,\mu)=(0,0)} = \sum_{j=0}^{N_*} \binom{N_*}{j} \frac{\partial^{N_*-j+1}g}{\partial \mu_i \partial t^{N_*-j}}(0,0) \left(\frac{\partial^{j+1}\alpha}{\partial y^{j+1}}(0,0)\right) , \tag{196}$$

exactly as in the map case, except now with $c = 1$.

On the other hand, the definition of ξ_{N_*} combined with (172), an identity that is exactly the same as its analogous in the proof of the map case, allows us to conclude, as we did then, that

$$\begin{aligned} &\frac{\partial F}{\partial x}(\xi_{N_*}(y,0),0) \frac{\partial \xi_{N_*}}{\partial \mu_i}(0,0) + \frac{\partial F}{\partial \mu_i}(\xi_{N_*}(y,0),0) \\ &\quad = M(\xi_{N_*}(y,0)) \begin{bmatrix} \delta_i p_{N_*}(y) \\ \delta_i q_{N_*}(y) \end{bmatrix} + R_{N_*}(y) e_1 , \end{aligned} \tag{197}$$

where $\delta_i p_{N_*}$, $\delta_i q_{N_*}$, and R_{N_*} are exactly as we have defined in (121).

In particular, $R_{N_*}^{(j)}(0) = 0$ and (124) and (125) hold here as well. Proceeding exactly as in the proof of Lemma 8.2, we prove that (134) is equivalent to

$$\begin{aligned} &\frac{\partial F}{\partial x}(0,0) \left(\frac{\partial^{N_*+1}\alpha}{\partial \mu_i \partial t^{N_*}}(0,0) - \frac{\partial^{N_*+1}\xi_{N_*}}{\partial \mu_i \partial t^{N_*}}(0,0)\right) = \\ &\quad \frac{\partial \alpha}{\partial t}(0,0) \frac{\partial^{N_*+1}g}{\partial \mu_i \partial t^{N_*}}(0,0) + \sum_{l=0}^{N_*-1} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Y_l^{N_*} - \sum_{l=0}^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) Z_l^{N_*} , \end{aligned} \tag{198}$$

where $Y_l^{N_*}$ and $Z_l^{N_*}$ are defined as in that proof.

The two statements of the induction process are now proved as in the map case. Multiplying (198) on the left by w^T and rearranging, it follows that

$$\frac{\partial^{N_*+1}g}{\partial \mu_i \partial^{N_*}}(0,0) = \sum_{l=0}^{N_*} \nu_l^{N_*} \frac{\partial \eta_{k-l}}{\partial \mu_i}(0) , \tag{199}$$

with

$$\nu_l^{N_*} := \frac{1}{b(w^T v)} \left(w^T Z_l^{N_*} - w^T Y_l^{N_*} \right) \ . \tag{200}$$

Also, $\nu_{N_*}^{N_*} \neq 0$ because, as proved in the map case,

$$\nu_{N_*}^{N_*} = (N_*)!(p_0'(0))^{N_*} \nu_0^0 \neq 0 \tag{201}$$

The second part of the induction, which consists in constructing p_{N_*+1}, q_{N_*+1}, and ξ_{N_*+1}, and the ensuing proof of of item (*iii*) are entirely analogous to the map case, and are thus omitted. □

9 Closing remarks

Underlying catastrophes were proposed as a methodological tool in [8, 9], a point of access to bifurcation theory in applications like the reaction-diffusion models in [1]. We have shown here that this concept actually goes further by providing, at least in the case of center dimension 1, a bridge between catastrophes and classical bifurcation theory.

The $\mathcal{B}$-$\mathcal{G}$ conditions were used to define the underlying catastrophes in [8], but here a more rigorous definition has been given, while the $\mathcal{B}$-$\mathcal{G}$ conditions provide a means to locate them. Together they yield a more practical approach to applying bifurcation theory in more diverse models of physical, biological or chemical phenomena. The new theoretical framework using $\mathcal{K}$-equivalence also provides a firm foundation for expanding the theory of underlying catastrophes to singularities whose corank is higher than 1, in forthcoming work.

As mentioned in Section 1, a remark on the case of maps Π with one single eigenvalue equal to -1 is merited. In this case, no singularity occurs in the displacement function f of Π, rendering the analysis presented in this paper inadequate. However, an underlying catastrophe germ does appear in the displacement of the iterated map Π^2. Thus, it should also be possible to analyze the 'flip' and other bifurcations via catastrophe theory, restricting the problem to a particular space of germs that are displacements of higher iterates of diffeomorphisms. The study of this is still in development.

A Auxiliary results

In this section, we present technical results that are needed in the course of the proofs of the theorems obtained in this paper.

A.1 Catastrophes of minimal stable families

The first auxiliary lemma connects the definition of minimal topologically stable families with the concept of underlying catastrophes. Its proof relies on the notion of $\mathcal{K}$-versal (or contact versal) unfoldings of a germ, an explanation of which can be found, for instance, in [14, Chapter 14]. In addition to the conditions for $\mathcal{K}$-versality that are available in this reference, we will be making use of the following classic result, the proof of which we omit (a stronger statement is proved, for instance, in [6, Theorem 7.4, Chapter XV]).

Proposition (Criterion for $\mathcal{K}$-equivalence)**.** *Two families with the same number of parameters and $\mathcal{K}$-equivalent critical germs, and which are each $\mathcal{K}$-versal unfoldings of these germs, are themselves $\mathcal{K}$-equivalent families.*

Lemma A.1. *Let $F(x, \mu)$ be a local k-parameter family of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ at $(0, 0)$. Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and suppose that $F(x, \mu) = (S(x_1, \mu), x_2)$ and S satisfies items (V.I$_k$) to (V.III$_k$). Then F undergoes an underlying catastrophe of corank 1 and codimension k at $x = 0$ for $\mu = 0$.*

Proof. Taking into account items (V.I$_k$) and (V.III$_k$), we can apply Taylor's theorem and conclude that there is a smooth function $r(x_1)$ such that $r(0) \neq 0$ and $S(x_1, 0) = x_1^{k+1} r(x_1)$. Hence, the germ of $s : x_1 \mapsto S(x_1, 0)$ at $x_1 = 0$ is $\mathcal{K}$-equivalent to $s^1_{1^k,0} : x_1 \mapsto x_1^{k+1}$, because $[s] = [r] \cdot [s^1_{1^k,0}]$.

We will prove that the k-parameter unfolding $[\tilde{S}]$ of $[s]$ given by $\tilde{S}(x_1, \mu) = (S(x_1, \mu), \mu)$ is $\mathcal{K}$-versal. The extended tangent space of $[s]$ is given by

$$T_{\mathcal{K},e}\, s = \{[s'] \cdot [X] + [M] \cdot [s] : X, M \in \mathcal{E}^1_1\} = \{[x^k q(x)] : q \in C^\infty(\mathbb{R})\} \tag{202}$$

Let $\mu = (\mu_1, \ldots, \mu_k)$. We shall prove that

$$T_{\mathcal{K},e}\, s \oplus \operatorname{span}\left(\frac{\partial S}{\partial \mu_1}(x, 0), \cdots, \frac{\partial S}{\partial \mu_k}(x, 0)\right) = \mathcal{E}^1_1 \,, \tag{203}$$

so that $[\tilde{S}]$ is $\mathcal{K}$-versal.

Considering (V.II$_k$), it follows that the matrix

$$\left[\begin{array}{c|ccc} \frac{\partial S}{\partial x_1}(0,0) & \frac{\partial S}{\partial \mu_1}(0,0) & \cdots & \frac{\partial S}{\partial \mu_k}(0,0) \\ \frac{\partial^2 S}{\partial x_1^2}(0,0) & \frac{\partial^2 S}{\partial \mu_1 \partial x_1}(0,0) & \cdots & \frac{\partial^2 S}{\partial \mu_k \partial x_1}(0,0) \\ \vdots & \vdots & & \vdots \\ \frac{\partial^{k+1} S}{\partial x_1^{k+1}}(0,0) & \frac{\partial^{k+1} S}{\partial \mu_1 \partial x_1^k}(0,0) & \cdots & \frac{\partial^{k+1} S}{\partial \mu_k \partial x_1^k}(0,0) \end{array}\right]$$

has non-zero determinant. Because of (V.I$_k$) and (V.III$_k$), this can only be if the matrix

$$A = (a_{ij}) := \begin{bmatrix} \frac{\partial S}{\partial \mu_1}(0,0) & \cdots & \frac{\partial S}{\partial \mu_k}(0,0) \\ \frac{\partial^2 S}{\partial \mu_1 \partial x_1}(0,0) & \cdots & \frac{\partial^2 S}{\partial \mu_k \partial x_1}(0,0) \\ \vdots & & \vdots \\ \frac{\partial^k S}{\partial \mu_1 \partial x_1^{k-1}}(0,0) & \cdots & \frac{\partial^k S}{\partial \mu_k \partial x_1^{k-1}}(0,0) \end{bmatrix} .$$

is invertible. This means that the inverse $A^{-1} := (b_{ij})^k_{i,j=1}$ is well defined, that is,

$$\delta_{i,j} = \sum_{l=1}^k a_{il} b_{lj} = \sum_{l=1}^k b_{lj} \frac{\partial S}{\partial \mu_l \partial x_1^{i-1}}(0,0) \,. \tag{204}$$

Observe that, by Taylor expansion, we obtain, for each $l \in \{1, 2, \ldots, k\}$, a smooth function $\rho_l(x_1)$ such that

$$\frac{\partial S}{\partial \mu_l}(x_1, 0) = \frac{\partial S}{\partial \mu_l}(0,0) + \frac{\partial S}{\partial \mu_l \partial x_1}(0,0) x_1 + \cdots + \frac{\partial S}{\partial \mu_l \partial x_1^{k-1}}(0,0) \frac{x_1^{k-1}}{(k-1)!} + \rho_l(x_1) x_1^k \,. \tag{205}$$

Hence, considering the definitions of $a_{ij}, b_{ij} \in \mathbb{R}$, it follows that, for each $j \in \{1, 2, \ldots, k\}$

$$\begin{aligned}\sum_{l=1}^{k} b_{lj} \frac{\partial S}{\partial \mu_l}(x_1, 0) &= \sum_{l=1}^{k}\sum_{i=1}^{k} b_{lj} \frac{\partial S}{\partial \mu_l \partial x_1^{i-1}}(0,0) \frac{x_1^{i-1}}{(i-1)!} + \sum_{l=1}^{k} b_{lj}\rho_l(x_1) x_1^k \\ &= \frac{x_1^{j-1}}{(j-1)!} + x_1^k \sum_{l=1}^{k} b_{lj}\rho_l(x_1) \ .\end{aligned} \tag{206}$$

Therefore, defining $\zeta_j(x_1) := (j-1)! \sum_{i=1}^{k} a_{lj}\rho_l(x_1)$, it follows that

$$\mathrm{span}\left(\frac{\partial S}{\partial \mu_1}(x,0), \cdots, \frac{\partial S}{\partial \mu_k}(x,0)\right) = \mathrm{span}\left(1 + x^k\zeta_1(x), x + x^k\zeta_2(x), \cdots, x^{k-1} + x^k\zeta_k(x)\right) \ . \tag{207}$$

It is then clear that

$$T_{\mathcal{K},e}\, s \oplus \mathrm{span}\left(\frac{\partial S}{\partial \mu_1}(x,0), \cdots, \frac{\partial S}{\partial \mu_k}(x,0)\right) = \mathcal{E}_1^1 \ , \tag{208}$$

and $[\tilde{S}]$ is a $\mathcal{K}$-versal unfolding of $[s]$.

Let

$$\mathcal{U}(y, \eta) = y^{k+1} + \eta_1 y^{k-1} + \eta_2 y^{k-2} + \ldots + \eta_k \ . \tag{209}$$

Considering that the unfolding $[\tilde{\mathcal{U}}^1]$ of $[s^1_{1^k,0}]$ given by $\tilde{\mathcal{U}}^1(y,\eta) = (\mathcal{U}(y,\eta), \eta)$ is also $\mathcal{K}$-versal and has k parameters, and also that $[s] = [r] \cdot [s^1_{1^k,0}]$, it follows that the families $[\tilde{\mathcal{U}}^1]$ and $[\tilde{S}]$ are $\mathcal{K}$-equivalent. Hence, there are smooth $Q(x_1, \mu) \in \mathbb{R}$, $\varphi(x, \mu) \in \mathbb{R}$ and $\eta(\mu) \in \mathbb{R}^k$ such that $Q^1(0,0) \neq 0$, $x \mapsto \varphi(x, 0)$ is a local diffeomorphism near zero, $\eta(0) = 0$, and

$$S(x_1, \mu) = Q^1(x_1, \mu) \cdot \mathcal{U}(\varphi(x_1, \mu), \eta(\mu)) \ . \tag{210}$$

Finally, by defining the local diffeomorphism $\psi(x_1, x_2, \mu) := (\varphi(x_1, \mu), x_2)$ and the smooth matrix function

$$Q(x_1, x_2, \mu) := \left[\begin{array}{c|c} Q^1(x_1, \mu) & 0 \\ \hline 0 & I_{n-1} \end{array}\right] \in \mathbb{R}^{n \times n} \ , \tag{211}$$

it follows that

$$F(x_1, x_2, \mu) = Q(x_1, x_2, \mu) \cdot \begin{bmatrix} \mathcal{U}(\psi_1(x_1, \mu), \eta(\mu)) \\ x_2 \end{bmatrix} \ , \tag{212}$$

and one can verify that this implies F satisfies both conditions of an underlying catastrophe of corank 1 at the origin for $\mu = 0$.

□

A.2 Properties of matrices with a simple zero eigenvalue

The second auxiliary result collects important properties of matrices with one simple zero eigenvalue and solutions of linear systems whose coefficient matrix is in that class.

Lemma A.2. *Let $A \in \mathbb{R}^{n\times n}$ be any matrix with a simple zero eigenvalue, associated to a left eigenvector w^T and a right eigenvector v. The following hold:*

(i) $v \in \mathbb{R}^n \setminus$ *Range*(A)*;*

(ii) *Range*$(A)^\perp = \langle w \rangle$*;*

(iii) *If* $v' \in \mathbb{R}^n \setminus$ *Range*(A)*, then* $w^T v' \neq 0$*. In particular,* $w^T v \neq 0$*.*

(iv) *There is a matrix* $E \in \mathbb{R}^{n\times n}$ *such that* $E\, v = 0$ *and, for any* $z \in$ *Range*(A)*, the solution set of the equation* $Ax = z$ *is* $\{Ez + rv : r \in \mathbb{R}\}$*.*

Proof. $v \in \mathbb{R}^n \setminus \mathrm{Range}(A)$, because otherwise there would be $v' \in \mathbb{R}^n \setminus \langle v \rangle$ such that $Av' = v \neq 0$, but

$$A^2\, v' = A\, v = 0\;, \tag{213}$$

which contradicts 0 being a simple eigenvalue of A. Thus, item (i) is proved.

Since $w^T A = 0$ and $\mathrm{rank}(A) = n-1$, it can only be that $\mathrm{Range}\,(A)^\perp = \langle w \rangle$, proving item (ii). Moreover, considering that

$$\mathbb{R}^n = \mathrm{Range}\,(A) \oplus \mathrm{Range}\,(A)^\perp = \mathrm{Range}\,(A) \oplus \langle w \rangle \tag{214}$$

and taking $v' \notin \mathrm{Range}\,(A)$, there are $v_0' \in \mathrm{Range}\,(A)$ and $b \neq 0$ such that $v' = v_0' + bw$. Therefore, $w^T \cdot v' = b\|w\|^2 \neq 0$, and item (iii) holds.

For the proof of item (iv), we invoke item (i) and the fact that $\mathrm{rank}(A) = n-1$ to justify splitting $\mathbb{R}^n$ as $\mathbb{R}^n = \mathrm{Range}(A) \oplus \langle v \rangle$. Then, define $\Pi \in \mathbb{R}^{n\times n}$ as the projection onto $\mathrm{Range}(A)$ with respect to this splitting. In particular, $\Pi\, v = 0$. Also, letting β be any basis of $\mathrm{Range}(A)$, we can construct the matrix $P \in \mathbb{R}^{(n-1)\times n}$ that represents the action of Π from the canonical basis to β, that is, if $x \in \mathbb{R}^n$, then $P\, x \in \mathbb{R}^{n-1}$ is equal to the coefficient vector representing $\Pi\, x \in \mathrm{Range}(A)$ in the basis β.

Let $A_R \in \mathbb{R}^{(n-1)\times(n-1)}$ be the matrix representing the action of A restricted to $\mathrm{Range}(A)$ on the basis β. In other words, for each $z \in \mathrm{Range}(A)$, let $[z] \in \mathbb{R}^{n-1}$ be its coefficient vector with respect to β. Then, $[Az] = A_R[z]$. It is clear that A_R must be invertible, otherwise there would be $z \in \mathrm{Range}(A)$ such that $Az = 0$, i.e., 0 would not be a simple eigenvalue of A. Thus, we are allowed to take its inverse A_R^{-1}. Finally, let Q be the matrix whose columns are the elements of β, so that $z = Q[z]$ for each $z \in \mathrm{Range}(A)$. Define the matrix E by

$$E := Q\, A_R^{-1}\, P. \tag{215}$$

To see that it satisfies the two properties stated in item (iv), notice first that, since $\Pi v = 0$, it follows that $P\, v = 0$ as well, so that $E\, v = 0$. Moreover, if $x \in \mathbb{R}^n$ and $z \in \mathrm{Range}(A)$ are such that $Ax = z$, then, by rewriting $x = \Pi x + (x - \Pi x) \in \mathrm{Range}(A) \oplus \langle v \rangle$ and noticing that $z = \Pi z$, it follows that $A\, \Pi\, x = \Pi z$.

Considering this equation in terms of the coefficients with respect to β, we obtain:

$$A_R P x = P z, \tag{216}$$

which, multiplied by the inverse of A_R, yields $[\Pi x] = Px = A_R^{-1} P z$, where $[\Pi x]$ are the coefficients of Πx with respect to β. Hence, multiplying both sides on the left by Q, it follows that

$$\Pi x = Q[\Pi x] = QPx = QA_R^{-1}Pz = Ez. \tag{217}$$

Taking into account once again that $x = \Pi x + (x - \Pi x) \in \text{Range}(A) \oplus \langle v \rangle$, it follows that there is $b \in \mathbb{R}$ such that $x = Ez + bv$. This proves that, if $x \in \mathbb{R}^n$ is a solution of $Ax = z$, then $x \in \{Ez + rv : r \in \mathbb{R}\}$.

The converse statement is proved as follows. Let $r \in \mathbb{R}$ and consider $x := Ez + rv$. Then, multiplying on the left by A, we obtain $Ax = AEz$, because $Av = 0$. Then, letting z' be the unique vector in Range(A) that satisfies $Az' = z$ and considering the definitions of E, P, and A_R, it follows that

$$AEz = AQA_R^{-1}Pz = AQA_R^{-1}[z] = AQ[z'] = Az' = z. \tag{218}$$

□

A.3 A fundamental lemma

Lemma A.3. *Let $p(t)$ be any real polynomial with vanishing constant coefficient, that is,*

$$p(t) = \sum_{i=1}^{N} a_i t^i \ , \tag{219}$$

with $N \in \mathbb{N}^*$ and $a_i \in \mathbb{R}$ for all $i \in \{1, \ldots, N\}$. Suppose $[f]$ is the germ of an underlying catastrophe of corank 1 and codimension k at $x = 0$, i.e. there are $[M] \in GL_n(\mathcal{E}_n)$ and $[\phi] \in L_n$ such that $[f] = [M] \cdot [s^n_{1^k}] \circ [\phi]$. Then for any representative ϕ of $[\phi]$, the curve $\gamma_p(t)$ defined by $\gamma_p(t) = \phi^{-1}(p(t), 0)$ satisfies

$$\frac{\partial^i}{\partial t^i} f(\gamma_p(t))\Big|_{t=0} = 0 \tag{220}$$

for all $i \in \{1, \ldots, k\}$.

Moreover,

$$f^{(k+1)}_{\gamma_p}(0) := \frac{\partial^{k+1}}{\partial t^{k+1}} f(\gamma_p(t))\Big|_{t=0} = (k+1)!\, a_1^{k+1}\, M(0)\, e_1 \ , \tag{221}$$

and, if $a_1 \neq 0$ and $\frac{\partial f}{\partial x}(0)$ has one simple eigenvalue equal to zero, all others having non-zero real part, then $w^T f^{(k+1)}_{\gamma_p}(0) \neq 0$, where w^T is the left eigenvector of $\frac{\partial f}{\partial x}(0)$ associated to zero.

Proof. Considering the hypotheses,

$$f(x) = M(x) \begin{bmatrix} (\phi_1(x))^{k+1} \\ \phi_2(x) \end{bmatrix} . \tag{222}$$

Therefore, it follows that

$$f(\gamma_p(t)) = M(\gamma_p(t)) \begin{bmatrix} (p(t))^{k+1} \\ 0 \end{bmatrix} = M(\gamma_p(t)) \begin{bmatrix} t^{k+1} \cdot q(t) \\ 0 \end{bmatrix} , \tag{223}$$

where $q(t)$ is also a real polynomial. In particular, it is clear, by differentiating both sides at $t = 0$, that

$$\frac{\partial^i}{\partial t^i} f(\gamma_p(t))\Big|_{t=0} = 0 \tag{224}$$

for all $i \in \{1, \dots, k\}$.

Furthermore, it is easy to see that $q(0) = a_1^{k+1}$. Hence,

$$\frac{\partial^{k+1}}{\partial t^{k+1}} f(\gamma_p(t))\Big|_{t=0} = M(0) \begin{bmatrix} (k+1)!\, q(0) \\ 0 \end{bmatrix} = (k+1)!\, a_1^{k+1} M(0)\, e_1 \ . \tag{225}$$

If $a_1 \neq 0$, it is clear that $\frac{\partial^{k+1}}{\partial t^{k+1}} f(\gamma_p(t)) \neq 0$ and, since

$$Df(0) = M(0) \begin{bmatrix} 0 \\ D\phi_2(0) \end{bmatrix} , \tag{226}$$

it follows that any element in the range of $Df(0)$ is a linear combination of all the columns of $M(0)$ except the first, a set that spans a $(n-1)$-dimensional space on account of the invertibility of $M(0)$. Considering that $Df(0)$ has corank 1, it follows that its range is equal to this span, so that the first column of $M(0)$ must fall outside it. Then by Lemma A.2, we conclude that $w^T f_{\gamma_p}^{(k+1)}(0) \neq 0$, as wanted. □

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