AVERAGING THEORY AND CATASTROPHES: THE PERSISTENCE OF BIFURCATIONS UNDER TIME-VARYING PERTURBATIONS

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ABSTRACT. When a dynamical system is subject to a periodic perturbation, averaging can be applied to obtain an autonomous leading order 'guiding system', placing the time dependence at higher order. We prove here that $\mathcal{K}$-universal bifurcations of the guiding system persist in the perturbed system, while the transcritical and pitchfork bifurcations do not, being instead perturbed into stable bifurcation families. We illustrate the results on examples of a fold, a transcritical, a pitchfork, and a saddle-focus, in systems with time-varying parameters.

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1. Introduction

The method of averaging essentially moves ε-sized time-varying terms in a dynamical system to higher orders in ε. This allows analysis of the time-independent leading order part, sometimes called the *guiding* system. Typically, if the perturbation is periodic, then equilibria of the guiding system constitute periodic orbits of the full system, see e.g. [16, 43] and, more recently and with more generality, [35, 39]. In fact, the method has also proved to be an efficient tool to investigate other invariant structures in non-autonomous dynamical systems, such as invariant tori [40, 38], and bifurcation types for families of systems depending on parameters [12, 16]. This paper fits in the programme of research outlined above, achieving a more general description of what can emerge from bifurcations in the guiding system.

In essence we will study here systems of the form $\dot{X} = \varepsilon f(t, X, \mu, \varepsilon)$, where $\mu \in \mathbb{R}^k$ is some parameter. According to standard results of averaging theory [43] we can write this as $\dot{x} = \varepsilon g(x, \mu) + \varepsilon^2 R(t, x, \mu, \varepsilon)$ after change of variables, so that the time-dependence enters only as a perturbation (albeit singular) of an otherwise autonomous system. Here f, g, R are differentiable functions we will specify more completely later.

Such equations have been around since Poincaré's study of systems of the form $\dot{u} = g(u) + \varepsilon h(u, t)$, from which are derived the origins of homoclinic tangles and chaos [41, 42, 18]. They remain novel today in multi-variable and multi-timescale problem, notably in models of neuron bursting via mixed-mode oscillations, see e.g. [23]. A simpler example is the singularly perturbed pendulum, $\ddot{u} = -\sin(u) + \varepsilon \sin(t/\varepsilon)$, e.g. [19]. These examples encapsulate the legacy of such studies, centering historically around the origins of chaos, and more recently around mixed-mode oscillations.

In contrast, what happens when such a system undergoes a bifurcation has received relatively little interest. In [16] it is shown that certain one-parameter bifurcations (a fold or a Hopf) of $\dot{x} = \varepsilon g(x, \mu)$ persist after being perturbed as above. Somewhat subtly, these theorems actually prove the existence of branches of equilibria or cycles around their bifurcation points, not of the bifurcations themselves, and moreover assume certain forms of system that are not entirely general. In [12], it is shown in more details that a Hopf bifurcation persists, resulting in a Neimark-Sacker bifurcation and creating invariant tori.

Here we will show indeed that a broad class of bifurcations of $\dot{x} = g_\ell(x, \mu)$ (in any number of parameters) persists under such perturbations. In fact we will show that many results follow quite simply from the central ideas of Thom's catastrophe theory, i.e., the use of singularity theory in the study of bifurcations of dynamical systems. We will use the idea from [20, 21, 22] of looking only at the *underlying catastrophes* of a system, which describe only bifurcations in numbers of equilibria, taking no interest in topological equivalence classes. This, for instance, makes no distinction between the germs $(\dot{x}, \dot{y}) = (x^2, y)$ and $(\dot{x}, \dot{y}) = (y, x^2)$, classifying them both has having an *underlying* fold at the origin, despite one involving a saddle and a node, the other involving a saddle and a focus. Unlike topological equivalence, however, this approach does distinguish between the germs $(\dot{x}, \dot{y}) = (x^2, y)$ and $(\dot{x}, \dot{y}) = (x^4, y)$ as they produce different numbers of equilibria under perturbation.

Our interest will particularly be in perturbations around non-hyperbolic points that induce bifurcations, such as a simple fold point with a small-time dependent perturbation, written as $\dot{x} = x^2 + \varepsilon^2 h(t)$. A notable application of this is to systems with time varying parameters. For example, consider the family $\dot{u} = u^2 + p$ depending on a parameter p.

What happens if, in fact, p undergoes small time fluctuations? We will show that the average value μ_p of $p(t)$ can play the role of a bifurcation parameter, the system being locally equivalent to $\dot{x} = \varepsilon(x^2 + \mu_p) + \varepsilon^2 \tilde{p}(t)$, which undergoes catastrophes of periodic solutions.

Using the same methods we will also show that certain bifurcations may not persist under perturbation, but may nevertheless form stable systems. In particular we will consider systems with transcritical and pitchfork bifurcations. These occupy a strange position in bifurcation theory, as they are not stable bifurcations in general, but are both important from the point of view of applications. In fact, the transcritical often occurs in systems (e.g. Lotka-Volterra) with trivial zeros, and the pitchfork in (though not exclusively) symmetric systems (e.g. something mechanical). Here we will show how, beginning with one of these non-generic bifurcations and adding the time-varying element, under perturbation these produce generic bifurcations of periodic solutions in the averaged system.

The paper is arranged as follows: in Section 2, we present an overview of the results obtained along with a brief presentation of the necessary terminology for their comprehension; in Section 4, we effectively introduce the concepts and known results extracted from both singularity theory and the averaging method which are needed to discuss and prove our results; finally, in Section 5, all the proofs of the results in this paper are presented, including a collection of auxiliary results of a more technical nature.

2. Overview of results

Consider a $(k+1)$-parameter family of n-dimensional systems in the form

$$\dot{X} = \sum_{i=1}^{N} \varepsilon^i F_i(t, X, \mu) + \varepsilon^{N+1} \tilde{F}(t, X, \mu, \varepsilon), \qquad (t, X, \mu, \varepsilon) \in \mathbb{R} \times D \times \Sigma \times (-\varepsilon_0, \varepsilon_0), \tag{1}$$

where D is an open, bounded neighbourhood of the origin in $\mathbb{R}^n$; Σ is an open, bounded neighbourhood of $\mu_* \in \mathbb{R}^k$; $\varepsilon_0 > 0$; $N \in \mathbb{N}^*$; and the functions F_i and $\tilde{F}$ are of class C^∞ in $\mathbb{R} \times D \times \Sigma \times (-\varepsilon_0, \varepsilon_0)$, and continuous and T-periodic in the variable t in $\mathbb{R} \times \overline{D} \times \overline{\Sigma} \times [-\varepsilon_0, \varepsilon_0]$.

We concern ourselves with T-periodic solutions of (1). Let $X(t, t_0, X_0, \mu, \varepsilon)$ be the solution of this system satisfying $X(t_0, t_0, X_0, \mu, \varepsilon) = X_0$. Suppose that the parameters (μ, ε) are fixed. To find T-periodic solutions, we will study the so-called stroboscopic Poincaré map Π, which is defined by

$$\Pi(X_0, \mu, \varepsilon) = x(T, 0, X_0, \mu, \varepsilon). \tag{2}$$

Since all functions present in the system are T-periodic, a fixed point of the Poincaré map corresponds to a T-periodic solution of (1).

If we allow the parameters to vary, different maps emerge, giving birth to a $(k+1)$-family of maps. In order to obtain a geometric picture of how the fixed points of Π change as the parameters (μ, ε) vary, we define the catastrophe surface M_Π of Π as the set of triples (X, μ, ε) such that $\Pi(X, \mu, \varepsilon) = X$.

Definition 1. *The catastrophe surface M_Π of the Poincaré map Π is defined by*

$$M_\Pi := \{(x, \mu, \varepsilon) \in D \times \Sigma \times (-\varepsilon_0, \varepsilon_0) : \Pi(x, \mu, \varepsilon) = x\}. \tag{3}$$

This definition is inspired by a similar concept appearing in Thom's catastrophe theory (see, for example, [8]). We remark that the term "surface" is used only for reasons of custom, and does not imply that M_Π is, globally or locally, a regular manifold in $D \times \Sigma \times (-\varepsilon_0, \varepsilon_0)$. We will see in Theorem 1 below that typically M_Π is not a manifold for the cases will be treating.

In this paper, we provide results locally characterizing the catastrophe surface of Π near bifurcation points for determinate classes of systems of the form (1). Crucially, the results we provide show that the knowledge of a special averaged form of the system - the so-called guiding system - is, in many instances, sufficient to fully describe M_Π. Essentially, we can infer in those cases that only the averaged effect of the time-dependent terms of (1) alter the qualitative behaviour of T-periodic solutions.

2.1. **Fibred maps.** Characterizing the catastrophe surface is not a matter of geometry only. In fact, take $n = k = 1$ and consider the maps $X \mapsto \Pi_1(X, \mu, \varepsilon) = X + X^2 - \mu$ and $X \mapsto \Pi_2(x, \mu, \varepsilon) = \mu^2$. The catastrophe surfaces of those maps are, respectively, given by $M_1 = \{(X, \mu, \varepsilon) : X^2 = \mu\}$ and $M_2 := \{(X, \mu, \varepsilon) : \mu^2 = X\}$. It is thus clear that M_2 can be obtained from M_1 from the rigid transformation of coordinates that rotates around the ε-axis by 90 degrees. Geometrically, thus, M_1 and M_2 are essentially identical.

However, the different roles played by the coordinate X and the parameters μ and ε make so that Π_1 undergoes a fold bifurcation at $\mu = 0$ and any $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, while Π_2 has exactly one fixed point $X^* = \mu^2$ for any pair (μ, ε). Hence, even though M_1 and M_2 are geometrically indistinguishable, the dynamics represented by them are certainly not equivalent.

This happens because the ambient space of M_Π is the product between the space of coordinates and the space of parameters. Thus, if we want a tool that guarantees that two systems are dynamically related by comparing their catastrophe surfaces, more than only geometric properties, we also need the difference between parameters and coordinates to be preserved. To do so, we introduce the concept of fibred maps.

Definition 2. *Let $U \subset D \times \Sigma \times (-\varepsilon_0, \varepsilon_0)$ be a neighbourhood of the origin. In the context established in this paper, a map $\Phi = (\Phi_1, \Phi_2, \Phi_3) : U \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}$ is said to be:*

- *weakly fibred if it is of the form $\Phi(x, \mu, \varepsilon) = (\Phi_1(x, \mu, \varepsilon), \Phi_2(\mu, \varepsilon), \Phi_3(\mu, \varepsilon))$;*
- *strongly fibred if it is of the form $\Phi(x, \mu, \varepsilon) = (\Phi_1(x, \mu, \varepsilon), \Phi_2(\mu, \varepsilon), \Phi_3(\varepsilon))$;*
- *weakly or strongly fibred to the m-th order at $p \in U$ if its m-jet at p is, respectively, weakly or strongly fibred.*

For instance, when considering M_1 and M_2 as above, it is clear that, even though those surfaces are geometrically identical, no fibred diffeomorphism exists taking one into the other. This can be seen as a consequence of the fact that fibred diffeomorphisms preserve the dynamical aspects of the catastrophe surface.

On the other hand, consider the map $X \mapsto \Pi_3(X, \mu, \varepsilon) = (X - \mu)^2 + X - \mu$, which still undergoes a fold bifurcation at $\mu = 0$. Its catastrophe surface is $M_3 = \{(X, \mu, \varepsilon) : (X - \mu)^2 = \mu\}$, and can be obtained by transforming M_1 via the fibred diffeomorphism $\Phi(X, \mu, \varepsilon) = (X + \mu, \mu, \varepsilon)$.

It is important to notice that, whilst weak fibration is sufficient to ensure the proper separation of coordinates and parameters, strong fibration is designed to tackle the specific conditions of applying the method of averaging to families, since we would like to keep track of the fact that the perturbation parameter ε is essentially distinguished from the other parameters, in that we will usually assume ε to be fixed and study the bifurcation family arising from varying μ.

Slightly altering the example we have already discussed, we consider the map $X \mapsto \Pi_4(X, \mu, \varepsilon) = (X - \mu + \varepsilon)^2 + X - \mu + \varepsilon$, which, for each fixed ε, still has a fold as we vary μ. Its catastrophe surface $M_4 = \{(X, \mu, \varepsilon) : (X - \mu + \varepsilon)^2 = \mu - \varepsilon\}$ is equal to the image of M_1 via the strongly fibred diffeomorphism $\Phi(X, \mu, \varepsilon) = (X + \mu, \mu + \varepsilon, \varepsilon)$.

It is easy to verify that the composition of two fibred maps is still fibred. It also holds that the inverse of a fibred diffeomorphism is itself fibred. These observations culminate in the following important result concerning the germs of fibred local diffeomorphisms, the proof of which can be found in appendix A.

Proposition 1. *The class of germs of weakly fibred local diffeomorphisms near the origin is a group with respect to composition of germs, and so is the class of germs of strongly fibred local diffeomorphisms near the origin.*

We remark that the concept of germs of local diffeomorphisms is introduced with more detail in Section 4.2.

2.2. **Averaging method and guiding system.** The averaging method allows us to simplify (1) by transforming it into a system that does not depend on time up to the N-th order of ε. More precisely, we are supplied with a smooth T-periodic change of variables $X \to x(t, X, \mu, \varepsilon)$ transforming (1) into

$$\dot{x} = \sum_{i=1}^{N} \varepsilon^i g_i(x, \mu) + \varepsilon^{N+1} r_N(t, x, \mu, \varepsilon), \tag{4}$$

where r_N is T-periodic in T. The periodicity of this change of variables allows us to conclude that T-periodic solutions of (4) correspond one-to-one with T-periodic solutions of (1).

It is also useful to remark that the change of variables provided by the averaging method is the identity for $t = 0$, so that M_Π can be identified with the catastrophe surface of the stroboscopic Poincaré map of (4). Henceforth, we will always take into account this identification, since as a rule we will be analysing (4) instead of (1) directly.

Further details about the transformation taking (1) into (4) will be provided in Section 4.5. At the moment, we limit ourselves to presenting a brief overview of which elements of (4) will be used to deduce general properties of the catastrophe surface.

Accordingly, assume that at least one of the elements of $\{g_1, \ldots, g_{N-1}\}$ is non-zero and let $\ell \in \{1, \ldots, N-1\}$ be the first positive integer for which g_ℓ does not vanish identically. Then, (4) can be rewritten as

$$\dot{x} = \varepsilon^\ell g_\ell(x, \mu) + \varepsilon^{\ell+1} R_\ell(t, x, \mu, \varepsilon), \tag{5}$$

where

$$R_\ell(t, x, \mu, \varepsilon) = \sum_{j=0}^{N-\ell-1} \varepsilon^j g_{j+\ell+1}(x, \mu) + \varepsilon^{N+1} r_N(t, x, \mu, \varepsilon). \tag{6}$$

System $\dot{x} = g_\ell(x, \mu)$, obtained by truncating (5) at the ℓ-th order of ε and rescaling time, is called the *guiding system* of (1). Our general approach will be to infer properties of M_Π from the singularity type appearing in the guiding system.

It is illustrative to sketch how ℓ and g_ℓ may be obtained from (1). First, we obtain g_1 by

$$g_1(x, \mu) = \frac{1}{T} \int_0^T F_1(t, x, \mu) dt, \tag{7}$$

the average of F_1 over $t \in [0, T]$. If g_1 does not vanish identically, then $\ell = 1$ and we are done. However, if $g_1 = 0$, we proceed similarly, defining g_2 to be the average of an expression involving the functions F_1 and F_2 over $t \in [0, T]$. Once again, we have to check whether $g_2 = 0$. If not, $\ell = 2$ and we are done, otherwise we have to continue in the same fashion. We do so until we reach the first g_ℓ that does not vanish identically. The expressions used to calculate the functions g_i and other details about the transformation of variables taking (1) into (4) will be provided in Section 4.5.

2.3. **Statement of the main result.** A celebrated result of the averaging method is that, if the guiding system has a simple equilibrium, then (1) has a T-periodic orbit for small ε (see [43]). Complementarily, our main result concerns the case when

$$\dot{x} = g_\ell(x, \mu) \tag{8}$$

has a singular equilibrium point at the origin for μ equal to a critical value μ_*. Without loss of generality, we assume that $\mu_* = 0$, that is,

(H1) $g_\ell(0,0) = 0$;
(H2) $\det\left(\frac{\partial g_\ell}{\partial x}(0,0)\right) = 0$.

In that case, we can state the following general result, which assumes that the family $g_\ell(x,\mu)$ containing the singular equilibrium is $\mathcal{K}$-universal, that is, "stable" in the sense of contact or $\mathcal{K}$-equivalence. More details about this concept, which is very useful in singularity theory (see, for instance, [15, 30, 33]), will be given in Section 4.2

Theorem 1. *Let $\dot{x} = g_\ell(x,\mu)$ be the guiding system associated to* (1)*, and assume that the vector field $x \mapsto g_\ell(x,0)$ has a singular equilibrium at the origin, i.e., items (H1) and (H2) hold. If the germ of $x \mapsto g_\ell(x,0)$ at zero has finite codimension and the k-parameter family $(x,\mu) \mapsto g_\ell(x,\mu)$ is a $\mathcal{K}$-universal unfolding of this germ, then there are neighbourhoods $U, V \subset \mathbb{R}^{n+k+1}$ of the origin and a strongly fibred diffeomorphism $\Phi : U \to V$ such that $\Phi(x,0,0) = (x,0,0)$ and the catastrophe surface M_Π of the family of Poincaré maps $\Pi(x,\mu,\varepsilon)$ satisfies*

$$M_\Pi \cap V = \Phi\left((Z_{g_\ell} \times \mathbb{R}) \cap U\right) \cup V_{\varepsilon=0}, \tag{9}$$

where $Z_{g_\ell} = \{(x,\mu) \in \mathbb{R}^{n+k} : g_\ell(x,\mu) = 0\}$ and $V_{\varepsilon=0} := \{(X,\mu,0) \in V\}$. Additionally, the set $Z_{g_\ell} \times \{0\}$ is invariant under Φ.

Observe that the set $Z_{g_\ell} \times \mathbb{R}$ appearing in the theorem is the catastrophe surface of the Poincaré map of the extended guiding system $\dot{x} = g_\ell(x,\mu)$, $\dot{t} = 1$. Hence, we are stating that M_Π consists in the union of two sets: a trivial part corresponding to $\varepsilon = 0$, since every point is a fixed point of Π in that case; and a non-trivial part that is, near the origin, the image under a strongly fibred diffeomorphism of the catastrophe surface of the extended guiding system.

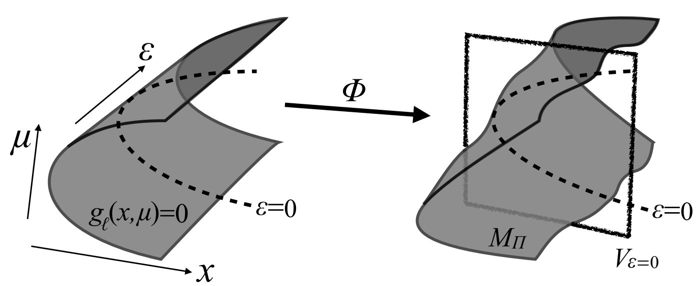

FIGURE 1. The catastrophe surface Z_{g_l} of the guiding system (left, suspended through $\varepsilon \in \mathbb{R}$), and the catastrophe surface M_Π of the time-dependent system (right). M_Π is the image of Z_{g_l} under the diffeomorphism Φ, and $Z_{g_l} \times \{0\}$ is invariant under Φ.

2.4. **Persistence of bifurcation diagrams for stable families.** A specially illustrative way to look at Theorem 1 is as ensuring the persistence of the well-known bifurcation diagrams of fixed points for $\mathcal{K}$-universal (also known as stable) families, as we will explain in this section.

For a general family of vector-fields $\dot{x} = F(x,\eta)$, the bifurcation diagram of equilibria is the subset of the coordinate-parameter space defined by $\{(x,\eta) : F(x,\eta) = 0\}$. Analogously, for a general family of maps $(x,\eta) \mapsto P(x,\eta)$, the bifurcation diagram of fixed points is $\{(x,\eta) : P(x,\eta) = x\}$.

In regard to the averaging method, the guiding system can be seen as the first non-trivial approximation of a system. It is thus desirable to determine to which degree this approximation allows us to extrapolate qualitative properties to the original system.

In the case treated in this paper, the guiding system is actually a family of vector fields undergoing some local bifurcation. The original system (1), however, has one extra perturbative parameter ε and is non-autonomous, so that the manner of comparison of its qualitative properties with those of the guiding system is not obvious.

To make this comparison possible, we fix $\varepsilon \neq 0$ small and compare the bifurcation diagrams of $\dot{x} = g_\ell(x,\mu)$ and $(x,\mu) \mapsto \Pi(x,\mu,\varepsilon)$, that is, we see the parameter ε as gauging a "perturbation" of the bifurcation diagram of the guiding system. We can then reinterpret Theorem 1 as stating that, for $\mathcal{K}$-universal families, the bifurcation diagrams of fixed points of the perturbed maps are actually $\mathcal{O}(\varepsilon)$ perturbations of the bifurcation diagram of equilibria of the guiding system, as follows.

Theorem 2. *With all the hypotheses and notation of Theorem 1, the bifurcation diagram* $\mathcal{D}_{\ell,0} := \{(x,\mu) \in D \times \Sigma : g_\ell(x,\mu) = 0\}$ *is locally a smooth manifold of codimension k near the origin. For* $\varepsilon \neq 0$ *sufficiently small, the perturbed bifurcation diagrams* $\mathcal{D}_\varepsilon := \{(x,\mu) \in D \times \Sigma : \Pi(x,\mu,\varepsilon) = x\}$ *are also smooth manifolds of codimension k near the origin, which are* $\mathcal{O}(\varepsilon)$*-close to* $\mathcal{D}_{\ell,0}$.

The proof of this result will be provided in Section 5.3. We would like to add that the above-mentioned extrapolation of qualitative properties can be stated as *persistence* of the bifurcation diagram from the guiding to the original system for small values of the perturbation parameter ε. The statement of the result in terms of persistence of qualitative properties of the guiding system is intended to mirror a selection of results in the area (see [43, Chapter 6] and, more recently, [12, 40].

2.5. **Stabilisation of non-stable families.** For non-stable families the bifurcation diagrams will not typically persist, as we show below for the transcritical and pitchfork bifurcations, instead they form stable families, in this case a pair of folds and a cusp, respectively.

2.5.1. *The transcritical.*

Theorem 3. *Let* $n = 1$ *and suppose that the guiding system* $\dot{x} = g_\ell(x,\mu)$ *undergoes a transcritical bifurcation at the origin for* $\mu = 0$*. If*

$$g_{\ell+1}(0,0) \neq 0, \tag{10}$$

then there are neighbourhoods $U, V \subset \mathbb{R}^{1+1+1}$ *of the origin and a strongly fibred diffeomorphism* $\Phi : U \to V$ *such that the catastrophe surface* M_Π *of the family of Poincaré maps* $\Pi(x,\mu,\varepsilon)$ *satisfies*

$$M_\Pi \cap V = \Phi\left(\{(y,\theta,\eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \eta = y^2 - \theta^2\} \cap U\right) \cup V_{\varepsilon=0}, \tag{11}$$

where $V_{\varepsilon=0} := \{(X,\mu,0) \in U\}$*. Additionally,* $\Phi(0,0,0) = (0,0,0)$,

$$\operatorname{sign}\left(\Phi_3'(0)\right) = \operatorname{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0,0)\right) \cdot \operatorname{sign}\left(g_{\ell+1}(0,0)\right), \tag{12}$$

and

$$(Z_{g_\ell} \times \{0\}) \cap V = \Phi\left(\{(y,\theta,0) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : y^2 - \theta^2 = 0\} \cap U\right). \tag{13}$$

2.5.2. *The pitchfork.*

Theorem 4. *Let $n = 1$ and suppose that the guiding system $\dot{x} = g_\ell(x,\mu)$ undergoes a pitchfork bifurcation at the origin for $\mu = 0$. If*

$$g_{\ell+1}(0,0) \neq 0, \tag{14}$$

then there are neighbourhoods $U, V \subset \mathbb{R}^{n+1+1}$ of the origin and a weakly fibred diffeomorphism $\Phi : U \to V$ such that the catastrophe surface M_Π of the family of Poincaré maps $\Pi(x,\mu,\varepsilon)$ satisfies

$$M_\Pi \cap V = \Phi\left(\{(y,\theta,\eta) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} : y^3 + y\theta + \eta = 0\} \cap U\right) \cup V_{\varepsilon=0}, \tag{15}$$

where $V_{\varepsilon=0} := \{(X,\mu,0) \in U\}$. Additionally, $\Phi(0,0,0) = (0,0,0)$, and Φ is strongly fibred to the first order at the origin.

2.6. **A discussion on topological equivalence.** Generally speaking, the catastrophe surface alone does note determine the topological class of the Poincaré map: there are potentially multiple topological classes with the same catastrophe surface. However, knowing the catastrophe surface reduces the number of possibilities for the topological types of the map, and we can see it as one of the elements constituting a general topological description.

We briefly explore this distinction by presenting the saddle-node case in one dimension, for which the catastrophe surface allows us to very easily infer topological conjugacy, and also exhibiting an interesting counter-example for planar vector fields.

2.6.1. *The one-dimensional case.* In the case of well-studied one-dimensional stable bifurcations, we can assert the topological conjugacy class of Π_ε by combining the strategy of unfolding the singularity of the germ of $x \mapsto \Delta_\ell(x,0,0)$ with known genericity conditions ensuring topological conjugacy to the normal form for the bifurcation (see [25, Theorems 4.1 and 4.2]). We provide an example of the application this method for the saddle-node bifurcation below.

Theorem 5. *Let $n = 1$ and suppose that the guiding system $\dot{x} = g_\ell(x,\mu)$ undergoes a saddle-node bifurcation at $(0,0)$, that is, assume that the following conditions hold:*

- *(F1)* $\frac{\partial g_\ell}{\partial \mu}(0,0) \neq 0$;
- *(F2)* $\frac{\partial^2 g_\ell}{\partial x^2}(0,0) \neq 0$.

Then, there are $\varepsilon_1 \in (0,\varepsilon_0)$, and smooth functions $x^* : (-\varepsilon_1,\varepsilon_1) \to D$ and $\mu^* : (-\varepsilon_1,\varepsilon_1) \to \Sigma$ such that:

- *(i)* $(x^*(0),\mu^*(0)) = (0,0)$.
- *(ii) For each $\varepsilon \in (-\varepsilon_1,\varepsilon_1)\setminus\{0\}$, the family of stroboscopic Poincaré maps $(x,\mu) \mapsto \Pi_\varepsilon(x,\mu)$ is locally topologically conjugate near $(x^*(\varepsilon),\mu^*(\varepsilon))$ to one of two possible normal forms: $(y,\beta) \mapsto (\beta - \mu^*(\varepsilon)) + (y - x^*(\varepsilon)) \pm (y - x^*(\varepsilon))^2$. In other words, the family of $(x,\mu) \mapsto \Pi_\varepsilon(x,\mu)$ is, up to translation of coordinates, locally topologically conjugate to one of the two topological normal forms for the saddle-node bifurcation for maps: $(y,\beta) \mapsto \beta + y \pm y^2$.*

We remark that an analogous result can be obtained for the cusp bifurcation, by considering the conditions available in [25, Theorem 9.1].

Proof. Let Φ be as in Theorem 1 and $\varepsilon \neq 0$ be small enough so that $\varepsilon' = \Phi_3^{-1}(\varepsilon)$ is well defined. By Theorem 1, the points (x,μ) near $(0,0)$ for which $\Delta_\ell(x,\mu,\varepsilon) = 0$ are given by $(\Phi_1(\alpha(t),\eta(t),\varepsilon'),\Phi_2(\eta(t),\varepsilon'))$, where $(\alpha(t),\eta(t))$ are a local parametrisation near $(0,0)$ of the curve given by $g_\ell(\alpha,\eta) = 0$. Considering the Implicit Function Theorem and items (F1) and (F2), we can assume that $\alpha(t) = t$.

Thus, since $g_\ell(t,\eta(t)) = 0$, by differentiating with respect to t at $t = 0$, we obtain

$$\frac{\partial g_\ell}{\partial \mu}(0,0)\eta'(0) = 0, \tag{16}$$

which ensures that $\eta'(0) = 0$.

Now, differentiating $\Delta_\ell(\Phi_1(t,\eta(t),\varepsilon'),\Phi_2(\eta(t),\varepsilon'),\varepsilon) = 0$ with respect to t at $t = 0$, it follows that

$$\frac{\partial \Delta_\ell}{\partial x}(\Phi_1(0,0,\varepsilon'),\Phi_2(0,\varepsilon'),\varepsilon) = 0. \tag{17}$$

Define $x^*(\varepsilon) := \Phi(0,0,\varepsilon')$ and $\mu^*(\varepsilon) := \Phi_2(0,\varepsilon')$. Since

$$\Pi_\varepsilon(x_0,\mu) = x_0 + \varepsilon^\ell \Delta_\ell(x_0,\mu,\varepsilon), \tag{18}$$

we obtain

$$\frac{\partial \Pi_\varepsilon}{\partial x_0}(x^*(\varepsilon),\mu^*(\varepsilon)) = 1 + \varepsilon^\ell \frac{\partial \Delta_\ell}{\partial x}(x^*(\varepsilon),\mu^*(\varepsilon),\varepsilon) = 1. \tag{19}$$

Taking into account [25, Theorems 4.1 and 4.2], we need only verify the two genericity conditions that guarantee a fold up to topological conjugacy:

($\mathrm{F_{top}}1$) $\frac{\partial \Pi_\varepsilon}{\partial \mu}(x^*(\varepsilon),\mu^*(\varepsilon)) \neq 0$;

($\mathrm{F_{top}}2$) $\frac{\partial^2 \Pi_\varepsilon}{\partial x^2}(x^*(\varepsilon),\mu^*(\varepsilon)) \neq 0$.

These follow directly from smoothness with respect to ε, from (18), and items (F1) and (F2). □

2.6.2. *The saddle-focus.* We looked at systems with *saddle-node* bifurcations in Section 2.6.1. The saddle-node is well known to be a generic one parameter bifurcation under topological equivalence, with normal form given by $(\dot{x}_1,\dot{x}_2,\dots,\dot{x}_n) = \left(x_1^2+\mu,\ x_2,\dots,x_n\right)$.

However, the collision between a saddle and a focus, which we will refer to as a saddle-focus, is not a generic one parameter bifurcation, but an example one parameter family is obtained if we interchange two entries on the right-hand side of the normal form family of the saddle-node: $(\dot{x}_1,\dot{x}_2,\dots,\dot{x}_n) = \left(x_2,\ x_1^2+\mu,\dots,x_n\right)$.

A generic family with a saddle-focus is found, for example, in the well studied Bogdanov-Takens bifurcation (see [1], [16], or [25]):

$$(\dot{x}_1,\dot{x}_2) = \left(x_2,\ x_1^2 \pm x_1x_2 + \lambda x_1^2 + \mu\right)\ , \tag{20}$$

requiring not just the parameter μ to unfold it, but also λ to control the local appearance of limit cycles and homoclinic connections. The singular germ corresponding to this family, obtained for zero values of the parameters, is $(\dot{x}_1,\dot{x}_2) = (x_2, x_1^2 + x_1x_2)$, and the two parameters appearing in the Bogdanov-Takens bifurcation ensure that this germ is of codimension 2 when considering topological equivalence.

However, it is interesting to notice that, if regarded as the germ of a plane map, the germ of $(x_1,x_2) \mapsto (x_2,x_1^2+x_1x_2)$ is actually $\mathcal{K}$-equivalent to $(x_1,x_2) \mapsto (x_2,x_1^2)$, which is itself equivalent to the saddle-node germ $(x_1,x_2) \mapsto (x_1^2,x_2)$. Essentially, this means that, with regards to the unfolding of zeroes of those map germs, i.e., equilibria of the corresponding vector fields, all three germs behave identically. Naturally, this observation does not allow us to obtain a complete description of the phase portrait of a family, as they are quite different, but the crucial point is that some of its elements can still be acquired.

In particular, we can describe the unfolding of equilibria of the germ of the vector field $(\dot{x}_1, \dot{x}_2) = (x_2, x_1^2)$, for which a complete unfolding with respect to topological equivalence is not known, and probably not even possible, hence the alternative germs unfolded in [3, 44] (the Bogdanov-Takens bifurcation) versus [13] (the Dumortier-Roussarie-Sotomayor bifurcation).

The latter of these provides a different generic family with a saddle-focus configuration,

$$(\dot{x}_1, \dot{x}_2) = \left(x_2,\ x_1^2 + \mu + x_2(\lambda_0 + \lambda_1 x_1 + x_1^3 x_2)\right) \ . \tag{21}$$

Like the Bogdanov-Takens bifurcation, this family requires not just the parameter μ to unfold it, but in this case two other parameters, λ_1 and λ_2. The singular germ of this family is $(\dot{x}_1, \dot{x}_2) = (x_2, x_1^2 + x_1^3 x_2^2)$. We will use this example to demonstrate to illustrate our results in Section 3.4.

3. The theory in practice: time-periodic coefficients

Before setting out the theory from Section 2 in detail, let us show how it works in practice on a few examples. For these we take the interesting applied problem of a system whose parameters are not exactly fixed, but vary slightly over time. To apply averaging we will assume that variation is periodic, for instance a physiological model in which some hormones have a small circadian perturbation, or a climate model where temperature has a small daily fluctuation.

3.1. **Example: persistence of fold catastrophe.**

Consider a system $\dot{Y} = Y^2$, perturbed by a parameter of order ε^2 and with a T-periodic fluctuation,

$$\dot{Y} = Y^2 + \varepsilon^2 f(t) \ . \tag{22}$$

For $\varepsilon \neq 0$, the change of variables $Y = \varepsilon X$ transforms this into

$$\dot{X} = \varepsilon\left(X^2 + f(t)\right), \tag{23}$$

which is a family of systems in the standard form (1).

If we define the average of $f(t)$ as

$$\mu := \frac{1}{T}\int_0^T f(t)dt \tag{24}$$

and the oscillating part of f to be $\tilde{f}(t) = f(t) - \mu$, we have

$$\dot{X} = \varepsilon\left(X^2 + \mu + \tilde{f}(t)\right), \tag{25}$$

where the average of $\tilde{f}$ over $[0, T]$ is zero. Accordingly, the stroboscopic Poincaré map of (25) will be denoted by Π and its catastrophe surface by M_Π.

Applying the transformation of variables given by the averaging theorem to obtain a system of the form (4), this family becomes

$$\dot{x} = \varepsilon(x^2 + \mu) + \varepsilon^2 R_1(t, x, \mu, \varepsilon). \tag{26}$$

It is then clear that the guiding system $\dot{x} = x^2 + \mu$ undergoes a fold bifurcation for $\mu = 0$. Theorem 1 ensures that M_Π locally has the form of a fold surface near the origin. Thus, we conclude that, for each small fixed $\varepsilon \neq 0$, a fold-like emergence (or collision) of fixed points of $x \mapsto \Pi(x,\mu,\varepsilon)$ occurs near 0 as μ traverses a neighbourhood of zero. The value of μ for which this occurs is given by a continuous function $\mu^*(\varepsilon)$ satisfying $\mu^*(0) = 0$.

As an example, let $f(t) = \mu + \sin(t)$, so

$$\dot{X} = \varepsilon(X^2 + \mu + \sin(t)). \tag{27}$$

While we cannot solve this exactly, it is instructive to look at its perturbative solution for small ε, which is $X(t, X_0, \mu, \varepsilon) \sim x_g(t, X_0, \mu, \varepsilon) + \varepsilon(1 - \cos t) + 2X_0\varepsilon^2(t - \sin t) + \mathcal{O}(\varepsilon^3)$, where x_g is the solution of the non-oscillatory problem $\dot{x} = \varepsilon(x^2 + \mu)$. Averaging this system amounts to removing the order ε oscillatory term by making a change of variables $X = x - \varepsilon \cos t$. In the method set out above, we have $\tilde{f}(t) = \sin(t)$ and $R_1(t, x, \mu, \varepsilon) = -2x\cos(t)$, giving the averaged system

$$\dot{x} = \varepsilon(x^2 + \mu) - 2\varepsilon^2 x \cos(t), \tag{28}$$

whose solutions satisfy $x(t, x_0, \mu, \varepsilon) \sim x_g(t, x_0, \mu, \varepsilon) - 2x_0\varepsilon^2 \sin t + \mathcal{O}(\varepsilon^3)$, where the oscillation has moved to higher order. The guiding system $\dot{x} = \varepsilon(x^2 + \mu)$ can be solved exactly, and its solutions are

$$\begin{aligned} x_g(t, x_0, \mu, \varepsilon) &= \sqrt{-\mu}\tanh\left(\varepsilon\sqrt{-\mu}t + \operatorname{arctanh}\left(\frac{x_0}{\sqrt{-\mu}}\right)\right) \\ &\sim x_0 + (x_0^2 + \mu)\varepsilon t + (x_0^2 + \mu)x_0\varepsilon^2 t^2 + \mathcal{O}(\varepsilon^3). \end{aligned} \tag{29}$$

These different solutions are illustrated in fig. 2, and we see the consequence of the results proven above, that for $\mu < 0$ the solutions of the exact and averaged systems all tend towards oscillation around the fixed points of the guiding system, but as μ moves to positive values a fold occurs and the fixed points vanish.

3.2. **Example: non-persistence of the transcritical bifurcation.** Consider the following differential system, with two time-dependent coefficients at different orders of ε:

$$\dot{Y} = Y^2 + \varepsilon f_1(t) Y + \varepsilon^2 f_2(t). \tag{30}$$

We assume f_1 and f_2 to be T-periodic. The change of variables $Y = \varepsilon X$ for $\varepsilon \neq 0$ yields

$$\dot{X} = \varepsilon\left(X^2 + f_1(t)X + \varepsilon f_2(t)\right). \tag{31}$$

Define the averages of f_1 and f_2 as

$$\mu := \frac{1}{T}\int_0^T f_1(t)dt, \qquad c := \frac{1}{T}\int_0^T f_2(t)dt, \tag{32}$$

and the oscillating part of f_1 as $\tilde{f}_1(t) := f_1(t) - \mu$. The system can then be rewritten as

$$\dot{X} = \varepsilon X^2 + \varepsilon\mu X + \varepsilon\tilde{f}_1(t)X + \varepsilon^2 f_2(t). \tag{33}$$

This is now in the form (1) with $N = 1$, $F_1(t, X, \mu) = X^2 + \mu X + \tilde{f}_1(t)X$, and $\tilde{F}(t, X, \mu, \varepsilon) = f_2(t)$. We then apply the change of variables given by the averaging theorem, that is, $X = x - \varepsilon(x + \varepsilon\cos t)\cos t$, obtaining

(34)
$$\dot{x} = \frac{\varepsilon}{(1 + \varepsilon A_1(t))}\left((x + \varepsilon x A_1(t))^2 + (x + \varepsilon x A_1(t))\mu + (x + \varepsilon x A_1(t))\tilde{f}_1(t) + \varepsilon f_2(t) - x A_1'(t)\right),$$

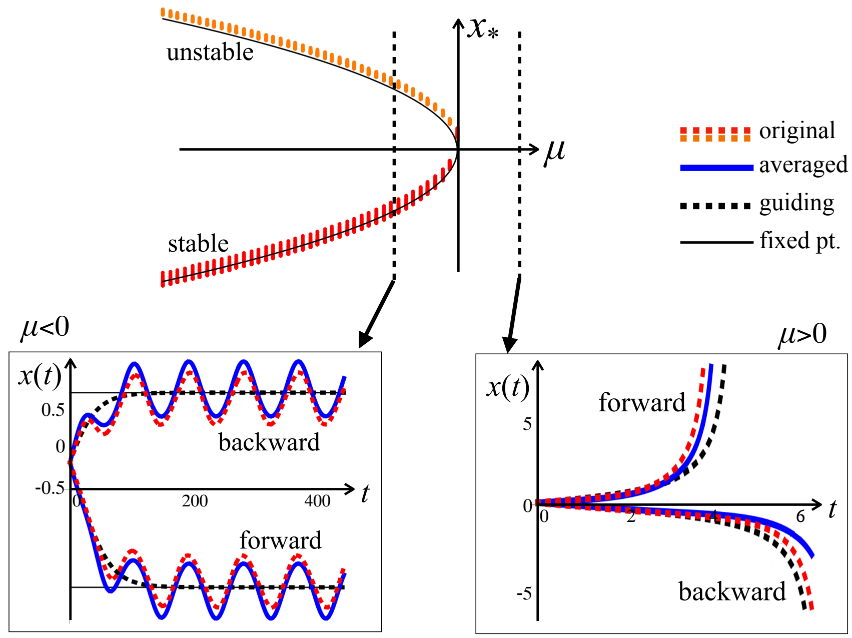

FIGURE 2. Solutions of the system (27) exhibiting a fold, with $\mu = \pm 0.5$ and $\varepsilon = 0.4$. The upper picture shows the Poincaré map of the original system (red/orange points), converging in forward/backward time to the fixed points of the guiding system (black curves). For two values of μ we plot the solutions below. For $\mu < \mu^*(0.5)$, the exact solutions (red dotted curves) and averaged solutions (blue curves) oscillate around the guiding solutions (black dotted curves), all converging in forward/backward time onto the stable/unstable fixed points (black lines). For $\mu > \mu^*(0.5)$ there are no fixed points and the solutions diverge. (For the exact solution we plot the variable $x = X + \varepsilon \cos t$).

where $A_1(t)$ is such that $A_1'(t) = \tilde{f}_1(t)$. Expanding in powers of ε, we obtain the averaged system:

$$\dot{x} = \varepsilon(x^2 + \mu x) + \varepsilon^2 \left(x^2 A_1(t) + x A_1(t) A_1'(t) + f_2(t) \right) + \mathcal{O}(\varepsilon^3). \tag{35}$$

It follows that the guiding system is $\dot{x} = g_1(x, \mu) = x^2 + \mu x$ and the remainder term is $R_1(t, x, \mu, \varepsilon) = x^2 A_1(t) + x A_1(t) A_1'(t) + f_2(t) + \mathcal{O}(\varepsilon)$. Thus,

$$g_2(0,0) = \int_0^T R_1(t, 0, 0, 0) dt = \int_0^T f_2(t) dt = c. \tag{36}$$

If $c \neq 0$, the system satisfies the hypotheses of Theorem 3, and the perturbation causes the described stabilisation of the catastrophe surface.

For illustration, let $f_1(t) = \mu + \sin(t)$ and $f_2(t) = c + \sin(2t)$, so we are studying the system $\dot{X} = \varepsilon(X^2 + X\mu + X \sin(t)) + \varepsilon^2(c + 2\sin(2t))$. Then $\tilde{f}_1(t) = \sin(t)$, $A_1(t) = -\cos(t)$, and $R_1(t, x, \mu, \varepsilon) = (c - (2 + x)\sin(t)\cos(t) - x^2 \cos(t))$. The averaging theorem yields the system $\dot{x} = \varepsilon(x^2 + \mu x) + \varepsilon^2(c - (2 + x)\sin(t)\cos(t) - x^2\cos(t))$.

Opposed to stable families, like the fold in Section 4.4, the catastrophe surface of the perturbed system will not lie close to that of the guiding system in the case of the transcritical. We can examine how the catastrophe surface unfolds with ε by taking the second order averaging system. We plot the zeros of this as an illustration of the catastrophe surface in fig. 3, which according to (45) approximates $M_\Pi - U_{\varepsilon=0}$ for small ε, and coincides with it at $\varepsilon = 0$.

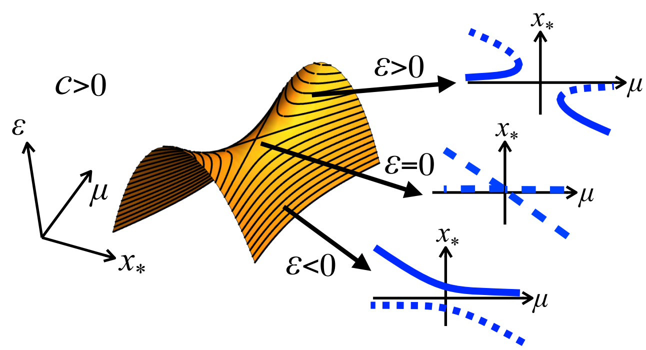

FIGURE 3. The surface of fixed points $x^2 + \mu x + \varepsilon c = 0$ plotted for $c = 1$ in (x, μ, ε) space (with x_* denoting the fixed-point value of x). The transcritical bifurcation at $\varepsilon = 0$ degenerates into a pair of fold bifurcations for $\varepsilon > 0$ and two stable families of fixed points for $\varepsilon < 0$. Sections of the surface at different ε give the bifurcation diagrams with varying μ.

Figure 4 shows simulations of Poincaré maps of the original system, which are a small perturbation of the bifurcation curves of the second order averaged system $\dot{x} = \varepsilon(x^2 + \mu x) + \varepsilon^2 c$, corresponding to the sections shown in fig. 3.

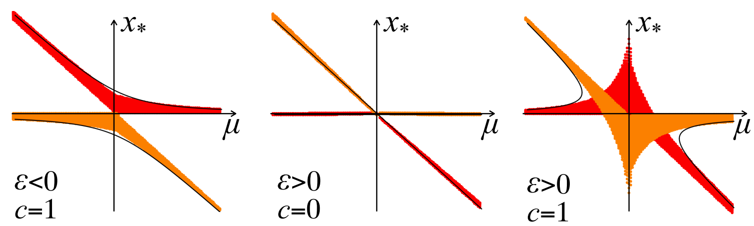

FIGURE 4. Solutions of the perturbed transcritical system. The Poincaré map of $x(t)$ mod $(t, 2\pi)$, showing exact solutions converging in forward time (red) and backward time (orange) onto the stable and unstable fixed points (black curves), respectively, from initial conditions close to the fixed points if they exist, or close to the origin otherwise. The parameters used are: left $\varepsilon = -0.02, c = 1$, middle $\varepsilon = 0.02, c = 0$, right $\varepsilon = 0.02, c = 1$. For $\varepsilon < 0$ there are always two fixed points. For $\varepsilon > 0$ and $c \neq 0$ there are two fixed points only for $|\mu| > \mu_{\text{fold}}$, so between the folds the solutions diverge.

Lastly, fig. 5 shows solutions for different values of μ and ε, showing the solutions converging onto a pair of fixed points, except for parameters values that lie between the two folds at which no fixed points exist, so solutions diverge.

3.3. **Example: non-persistence of the pitchfork.** Similar to the example for the transcritical, consider a system with two T-periodic parameters at different orders of ε, but this time take

$$\dot{Y} = Y^3 + \varepsilon^2 f_1(t) Y + \varepsilon^4 f_2(t). \tag{37}$$

The change of variables $Y = \varepsilon X$ for $\varepsilon \neq 0$ yields

$$\dot{X} = \varepsilon^2 \left(X^3 + f_1(t) X + \varepsilon f_2(t) \right). \tag{38}$$

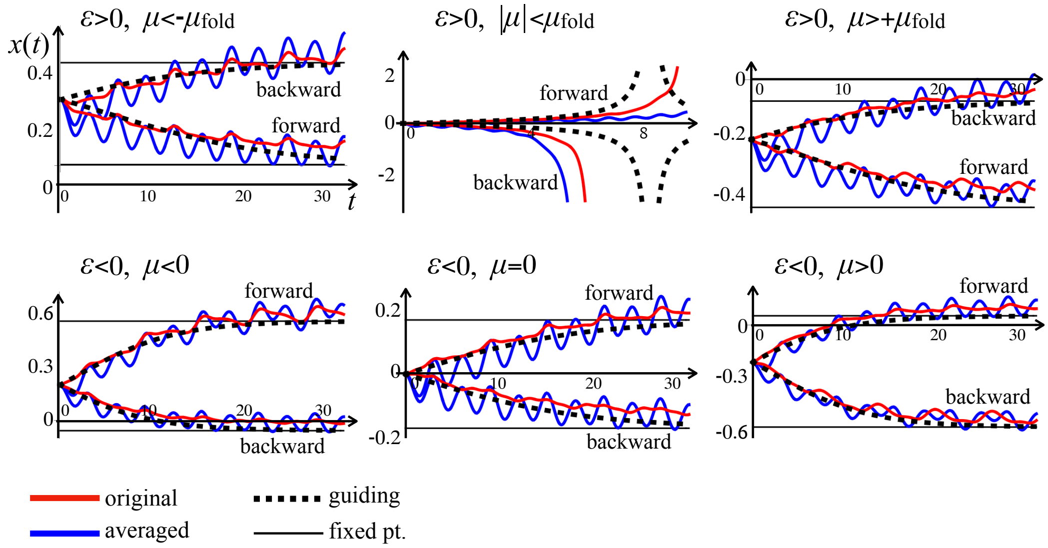

FIGURE 5. Solutions of the perturbed transcritical system with $c = 0.1$, for $\varepsilon = 0.3$ (top row) and $\mu = -0.3$ (bottom row), with values $\mu = -0.5, 0, 0.5$, (from left to right). The original solutions (red curves) and averaged solutions (blue curves) oscillate around the guiding solutions (black dotted curves), converging in forward/backward time onto the stable/unstable fixed points (blue curves) if they exist. For $\varepsilon > 0$ there are two fixed points only for $|\mu| > \mu_{\text{fold}}$, so in the middle picture the solutions diverge. For $\varepsilon < 0$ there are always two fixed points.

If μ denotes the average over $[0, T]$ of f_1 and $\tilde{f}_1(t) := f_1(t) - \mu$, we obtain

$$\dot{X} = \varepsilon^2 \left(X^3 + \mu X + \tilde{f}_1(t) X \right) + \varepsilon^3 f_2(t). \tag{39}$$

This system is in the standard form with $N = 2$, $F_1(t, X, \mu) = 0$, $F_2(t, X, \mu) = X^2 + \mu X + \tilde{f}_1(t) X$, and $\tilde{F}(t, x, \mu, \varepsilon) = f_2(t)$.

We can then apply the change of variables given by the averaging theorem. Let $A_1(t)$ be such that $A_1'(t) = \tilde{f}_1(t)$. We perform the change of variables given by $X = x + \varepsilon x A_1(t)$, obtaining

$$\dot{x} = \varepsilon^2 (x^3 + \mu x) + \varepsilon^3 \left(f_2(t) - A_1(t) \left(\mu x + x^3 \right) \right) + \mathcal{O}(\varepsilon^4). \tag{40}$$

Following the naming convention of the averaging method, we have the guiding system $\dot{x} = g_2(x, \mu) = x^3 + \mu x$ and $R_2(t, x, \mu, \varepsilon) = f_2(t) - A_1(t) \left(\mu x + x^3 \right) + \mathcal{O}(\varepsilon)$. Thus, assuming that the average of f_2 over $[0, T]$ does not vanish, it follows that

$$g_3(0, 0) = \int_0^T R_2(\tau, 0, 0, 0) d\tau = \int_0^T f_2(\tau) d\tau \neq 0. \tag{41}$$

We are therefore within the domain of application of Theorem 4.

For illustration, let $f_1(t) = \mu + \sin(t)$ and $f_2(t) = c + \sin(2t)$, so we are studying the system $\dot{X} = \varepsilon^2 (X^3 + \mu X + \sin(t) X) + \varepsilon^3 (c + 2 \sin(2t))$. Then $\tilde{f}_1(t) = \sin(t)$, and $R_1(t, x, \mu, \varepsilon) = (c - (2 + y) \sin(t) \cos(t) - 2y^3 \cos(t))$, hence the averaging theorem yields the system $\dot{y} = \varepsilon (y^3 + \mu y) + \varepsilon^2 (c - (2 + y) \sin(t) \cos(t) - 2y^3 \cos(t))$.

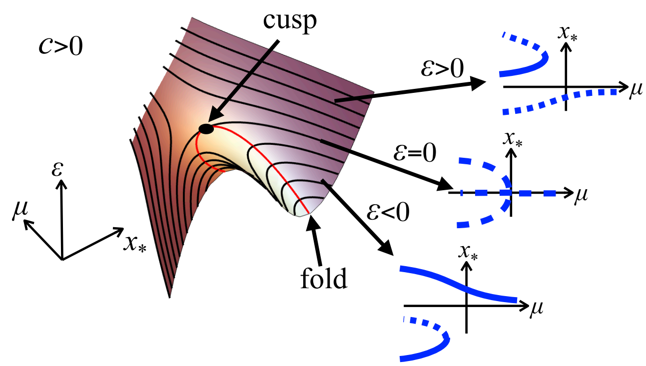

FIGURE 6. A cusp bifurcation appearing for $c = 1$. The fixed points are plotted in (x, μ, ε) space (with x_* denoting the fixed point value of x). Sections of this at different ε give the bifurcation diagrams with varying μ, showing a fold.

3.4. **A counter-example to topological equivalence: the saddle-focus.** Let us consider the following two-dimensional family:

$$(\dot{x}_1, \dot{x}_2) = \varepsilon \left(x_2,\ x_1^2 + \mu + \sin(t) \right) + \varepsilon^2 \left(0, x_2(c_0 + c_1 x_1 + x_1^3 x_2) \right) . \tag{42}$$

For the averaged system we obtain

$$(\dot{x}_1, \dot{x}_2) = \varepsilon \left(x_2,\ \mu + x_1^2 \right) + \varepsilon^2 \left(-\cos(t), x_2(c_0 + c_1 x_1 + x_1^3) \right) + \mathcal{O}(\varepsilon^3). \tag{43}$$

The guiding system $(\dot{x}_1, \dot{x}_2) = \varepsilon \left(x_2,\ \mu + x_1^2 \right)$ is structurally unstable.

The purpose of this example is to show that we do not need structural stability in the guiding system, or topological equivalence to the form of bifurcation we derive, hence this example can be analysed using our main theorem, but evidently we should not expect a full topological description of the map as in Section 2.6.1.

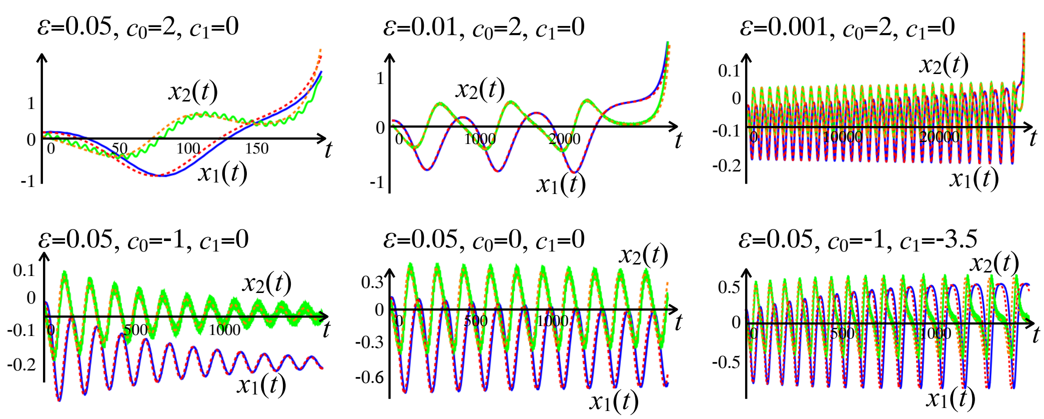

FIGURE 7. Solutions of the system exhibiting a saddle-focus. Initial conditions $(x_1, x_2) = (0.1, 0)$. In all cases we take $\mu = -0.2$ (as $\mu > 0$ merely gives diverging solutions). Other parameters as shown. Forward-time solutions shown only, showing the exact x_1/x_2, components in blue/green, and averaged system x_1, x_2, in dotted red/orange. Showing top: unstable systems with $c_0 = 2$, $c_1 = 1$; bottom: stabilised by different c_1, c_0, values.

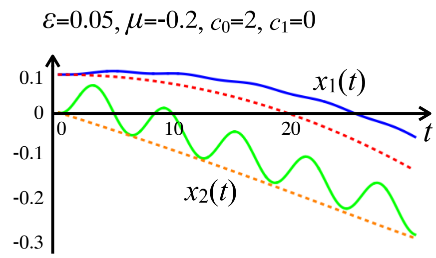

FIGURE 8. Zoom in on one of the solutions so we can see the difference between exact and averaged solution.

4. Preliminaries

In this section, we introduce the basic framework necessary to prove our results. The reason for this introduction is twofold: it should help the reader's comprehension, avoiding long detours in multiple different references; and it should also establish notation and nomenclature.

The material herein presented is already known, and, as a rule, we shall only refer the reader to references containing proofs when those are necessary. However, we still decided to provide our own proofs when the result is stated in such a specific manner that attempting to fetch the argument elsewhere may prove excessively inconvenient.

4.1. The Poincaré map and the displacement function of order ℓ.

The use of displacement functions to study the Poincaré map has numerous examples in the literature (see, for instance, [6, 7, 9, 11]). As we will see below, the problem of finding fixed points of the Poincaré map is then reduced to searching for zeroes of this function, and multiple approaches to accomplish this search can be developed depending on the hypotheses assumed to hold.

In this paper, we will study singular zeroes with the aid of singularity theory. As such, we always assume that the guiding system has a singular equilibrium point at the origin for $\mu = 0$, a condition which is detailed by items (H1) and (H2).

If $x(t, x_0, \mu, \varepsilon)$ denotes the solution of (4) satisfying $x(0, x_0, \mu, \varepsilon) = x_0$, then the family of stroboscopic Poincaré maps Π is given by $\Pi(x_0, \mu, \varepsilon) := x(T, x_0, \mu, \varepsilon)$. As mentioned, fixed points of Π correspond to T-periodic solutions of (4) and, consequently, (1).

We introduce the displacement function Δ:

$$\Delta(x_0, \mu, \varepsilon) = \Pi(x_0, \mu, \varepsilon) - x_0 = \varepsilon^\ell \int_0^t g_\ell(x(\tau, x_0, \mu, \varepsilon), \mu) + \varepsilon R_\ell(\tau, x(\tau, x_0, \mu, \varepsilon), \mu, \varepsilon) d\tau. \tag{44}$$

Thus, in looking for fixed points of Π, we focus on finding zeroes of the displacement function Δ. Since our interest lies essentially on the case $\varepsilon \neq 0$, we dispose of the ε^ℓ term in the displacement function to simplify the search, defining $\Delta_\ell : \tilde{D} \times \Sigma \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ by

$$\Delta_\ell(x_0, \mu, \varepsilon) = \int_0^t g_\ell(x(\tau, x_0, \mu, \varepsilon), \mu) + \varepsilon R_\ell(\tau, x(\tau, x_0, \mu, \varepsilon), \mu, \varepsilon) d\tau. \tag{45}$$

Throughout this paper, we will call Δ_ℓ the *displacement function of order* ℓ. The key step in our analysis is noticing that (45) guarantees that Δ_ℓ provides an unfolding of the germ of the map $x \mapsto g_\ell(x, 0)$. We will mostly work from this very observation throughout the rest of the paper. The notions of unfolding and germ will be formalized and explained later, in Section 4.2.

Observe that, by definition,

$$\Pi(x_0, \mu, \varepsilon) = x_0 + \varepsilon^\ell \Delta_\ell(x_0, \mu, \varepsilon). \tag{46}$$

This is an essential identity to the analysis done in this paper, since it provides the relation between the stroboscopic Poincaré map and the displacement function of order ℓ. Using this identity, we will establish relationships between important elements of each of those functions.

It is easy to establish from (46) that M_Π can be identified with the set $Z_{\Delta_\ell} \cup \{(x, \mu, 0) : (x, \mu) \in D \times \Sigma\}$, where Z_{Δ_ℓ} is the set of zeroes of Δ_ℓ. The study of M_Π can then be performed by analysing the zeroes of the displacement function of order ℓ.

4.1.1. *Fixed points and zeroes.* Proceeding with the general approach outlined above, we formalise the terms we will adopt when referring to and classifying zeroes of families of functions. We also prove the basic results connecting the fixed points of Π with the zeroes of Δ_ℓ.

Definition 3. *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and V an open subset of $\mathbb{R}^k$, $k \in \mathbb{N}$, and $P : U \times V \to \mathbb{R}^m$ be any family of functions. We define*

(a) the zero set of F by $Z_P = \{(x, \eta) \in U \times V : P(x, \eta) = 0\}$;
(b) the function $F_\eta : U \to \mathbb{R}^m$, where $\eta \in V$, by $P_\eta(x) = F(x, \eta)$;
(c) the zero set of F_η, where $\eta \in V$, by $Z_P(\eta) = \{x \in U : F(x, \eta) = 0\}$.

Any element of Z_P is called a zero of P, and any element of $Z_P(\eta)$ is called a zero of P_η.

The following results are the main point of this section, connecting fixed points of Π and zeroes of Δ_ℓ, as well as expressing M_Π with the help of the set of zeroes of Δ_ℓ. Their proofs follow directly from (46).

Proposition 2. *Let $\mu \in \Sigma$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ be given. Then, $x \in D$ is a fixed point of $x \mapsto \Pi(x, \mu, \varepsilon)$ if, and only if, (x, μ, ε) is a zero of Δ_ℓ.*

Corollary 1. $M_\Pi = Z_{\Delta_\ell} \cup \{(x, \mu, 0) : (x, \mu) \in D \times \Sigma\}$.

Proof. It suffices to notice that $\Pi(x, \mu, 0) = x$ for all $(x, \mu) \in D \times \Sigma$ and also consider Proposition 2. □

4.2. **Germs and $\mathcal{K}$-equivalence.**

4.2.1. *Germs.* An important concept in the study of bifurcations (and also in singularity theory) is that of *germs*. Essentially, a germ of an object captures only its only local properties. It is usually expressed as an equivalence class. We define that concept below, and set the notation we will be adopting throughout the remainder of the paper.

For convenience, we assume without loss of generality that the point near which our analysis is done will always be the origin, and we say that $U \in \mathcal{N}_0(S)$ if U is an open neighbourhood of the origin contained in the set S. The first two definitions build the concept of germs of maps.

Definition 4. *Let $n, p \in \mathbb{N}$ and $U, U' \in \mathcal{N}_0(\mathbb{R}^n)$. Two maps $f : U \to \mathbb{R}^p$ and $g : U' \to \mathbb{R}^p$ are said to be germ-equivalent at the origin if there is $U'' \in \mathcal{N}_0(U \cap U')$ such that $f|_{U''} = g|_{U''}$.*

Definition 5. *Let $n, p \in \mathbb{N}$ and $U \in \mathcal{N}_0(\mathbb{R}^n)$. The germ of a map $f : U \to \mathbb{R}^p$ at the origin is the equivalence class $[f]$ of f under germ-equivalence at the origin. The set of germs of functions from $\mathbb{R}^n$ to $\mathbb{R}^p$ at the origin is denoted by $\mathcal{E}_n^p$.*

The set of germs of maps has a useful algebraic structure when equipped with operations induced by the operations defined between maps, as stated in the following proposition, the proof of which is left to the reader.

Proposition 3. *$\mathcal{E}_n^p$ is a vector field over $\mathbb{R}$ with the naturally induced operations of sum of functions and product of a function by a real number.*

In singularity theory, it is usually useful to distinguish the special class of germs of diffeomorphisms:

Definition 6. *Let $n \in \mathbb{N}$. $[\phi] \in \mathcal{E}_n^n$ is said to be the germ of a local diffeomorphism at the origin if there is one element ϕ in the class $[\phi]$ for which*

(1) $\phi(0) = 0$;

(2) $\phi'(0)$ *is invertible.*

The set of germs of local diffeomorphisms at the origin on $\mathbb{R}^n$ *is denoted by* L_n.

Observe that L_n is a group under the natural operation induced by composition. Moreover, the group L_n acts on $\mathcal{E}_n^p$ on the right by the operation induced naturally by composition.

From now on, we adopt the notation $[f] : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ to mean that $[f] \in \mathcal{E}_n^p$ and that $f(0) = 0$. Equivalently, we may say that $[f] \in \mathcal{Z}_n^p$. We proceed now to the crucial concept of unfolding of a germ.

Definition 7. *A k-parameter unfolding of a germ* $[f] : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is a germ* $[\tilde{F}] \in (\mathbb{R}^{n+k}, 0) \to (\mathbb{R}^{p+k}, 0)$ *such that*

(1) *a representative* $\tilde{F}$ *of* $[\tilde{F}]$ *is of the form* $\tilde{F}(x, \eta) = (F(x, \eta), \eta)$;
(2) $F(x, 0) = F_0(x) = f(x)$.

The set of k-parameter unfoldings is denoted by $\mathcal{Z}_{n,k}^p$. *More specifically, the set of k-parameter unfoldings of the identity in* $\mathbb{R}^n$ *is denoted by* $L_{n,k}$.

4.2.2. *$\mathcal{K}$-equivalence.* The concept of $\mathcal{K}$-equivalence - also known as contact equivalence (see [15, 31, 33]) or V-equivalence (see [30]) - is part of the standard theory of singularities. We briefly introduce it here, confined to what is necessary to our discussion. The interested reader is referred to the more thorough presentation in [33].

Definition 8. *Two germs* $[f], [g] \in \mathcal{Z}_n^p$ *are said to be* ***$\mathcal{K}$-equivalent*** *if there are* $[\phi] \in L_n$ *and* $[M] \in GL_p(\mathcal{E}_n)$ *such that* $[f] = [M] \cdot [g] \circ [\phi]$.

The concept of $\mathcal{K}$-equivalence can also be used to study families of maps, through the ideas of $\mathcal{K}$-induction and $\mathcal{K}$-equivalent unfoldings, which are developed in the next definitions.

Definition 9. *Two unfoldings* $[\tilde{F}], [\tilde{G}] \in \mathcal{Z}_{n,k}^p$ *of the same germ* $[f] \in \mathcal{Z}_n^p$ *are said to be* ***$\mathcal{K}$-isomorphic*** *if, for any representatives* $\tilde{F}(x, \eta) = (F(x, \eta), \eta)$ *and* $\tilde{G}(x, \eta) = (G(x, \eta), \eta)$ *of* $[\tilde{F}]$ *and* $[\tilde{G}]$, *respectively, there are a smooth function* α, *defined on a neighbourhood* $U \times V$ *of the origin in* $\mathbb{R}^n \times \mathbb{R}^k$ *and satisfying* $\alpha(x, 0) = x$, *and a smooth matrix function* Q, *also defined on* $U \times V$ *and satisfying* $Q(x, 0) = I_p$, *such that the identity*

$$F(x, \eta) = Q(x, \eta) \cdot G(\alpha(x, \eta), \eta) \tag{47}$$

holds in $U \times V$.

Definition 10. *Let* $[\tilde{F}] \in \mathcal{Z}_{n,l}^p$ *and* $[h] : (\mathbb{R}^k, 0) \to (\mathbb{R}^l, 0)$. *The pullback of* $[\tilde{F}]$ *by* $[h]$, *denoted by* $[h]^*[\tilde{F}]$, *is the unfolding* $[\tilde{P}] \in \mathcal{Z}_{n,k}^p$ *given by*

$$\tilde{P}(x, \eta) = (F(x, h(\eta)), \eta). \tag{48}$$

Definition 11. *The unfolding* $[\tilde{F}] \in \mathcal{Z}_{n,k}^p$ *of the germ* $[f] \in \mathcal{Z}_n^p$ *is said to be* ***$\mathcal{K}$-induced*** *by the unfolding* $[\tilde{G}] \in \mathcal{Z}_{n,l}^p$ *of the same germ via* $[h] : (\mathbb{R}^k, 0) \to (\mathbb{R}^l, 0)$ *if the unfoldings* $[\tilde{F}]$ *and* $[h]^*[\tilde{G}]$ *are $\mathcal{K}$-isomorphic. In other words, if* $\tilde{F}(x, \eta) = (F(x, \eta), \eta)$, $\tilde{G}(x, \xi) = (G(x, \xi), \xi)$, *and* $h(\eta)$ *are representatives of* $[\tilde{F}]$, $[\tilde{G}]$, *and* $[h]$, *respectively, there are neighbourhoods of the origin* $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$, *and smooth functions* $\alpha : U \times V \to \mathbb{R}^n$ *and* $Q : U \times V \to \mathbb{R}^{p \times p}$ *such that* $h(0) = 0$, $\alpha(x, 0) = x$, $Q(x, 0) = I_p$, *and*

$$F(x, \eta) = Q(x, \eta) \cdot G(\alpha(x, \eta), h(\eta)) \tag{49}$$

for $(x, \eta) \in U \times V$.

Definition 12. *The unfolding $[\tilde{F}] \in \mathcal{Z}^p_{n,k}$ of the germ $[f] \in \mathcal{Z}^p_n$ is said to be **$\mathcal{K}$-equivalent** to the unfolding $[\tilde{G}] \in \mathcal{Z}^p_{n,k}$ of the same germ if there is $[h] \in L_k$ such that the unfolding $[\tilde{F}]$ is $\mathcal{K}$-induced by $[\tilde{G}]$ via $[h]$.*

4.2.3. *Versality and codimension.* One of the central concepts of singularity theory is that of versality. In essence, an unfolding of a germ of a map is said to be versal if it induces all other possible unfoldings of the same germ. This means that all the possible unfoldings of that germ are codified in a versal unfolding, so that a versal unfolding carries all the information needed to unfold a germ.

Naturally, different equivalence relations give rise to different notions of induction and equivalence, and thus also different notions of versality. In this paper, we are interested in $\mathcal{K}$-versality, i.e, versality with respect to $\mathcal{K}$-equivalence.

Definition 13. *The unfolding $[\tilde{F}] \in \mathcal{Z}^p_{n,k}$ of the germ $[f] \in \mathcal{Z}^p_n$ is said to be **$\mathcal{K}$-versal**, if any other unfolding $[\tilde{G}] \in \mathcal{Z}^p_{n,k}$ of $[f]$ is $\mathcal{K}$-induced by $[\tilde{F}]$ via the germ of some mapping $[h]$.*

The theory of singularities provides a number of results aimed at verifying $\mathcal{K}$-versality, one of the its products being the important concept of codimension of a germ. Below, we present those results and the concepts involved. With the exception of Proposition 7, for which we provide a proof on account of its specificity, all other proofs are well-known and can be found, for instance, in [33].

Definition 14. *Let $\boldsymbol{X}^0_n$ be the set of germs of n-dimensional vector fields at the origin having zero as equilibrium and $\boldsymbol{M}^0_p$ be the set of germs of matrix functions $\mathbb{R}^n \to \mathbb{R}^{p\times p}$ at the origin of $\mathbb{R}^n$. The extended $\mathcal{K}$-tangent space of a germ $[f] \in \mathcal{Z}^p_n$ is defined as the subspace of the vector field $\mathcal{E}^p_n$ given by:*

$$T_{\mathcal{K},e}f := \{[f'] \cdot [X] + [M] \cdot [f] : [X] \in \boldsymbol{X}^0_n, [M] \in \boldsymbol{M}^0_p\}. \tag{50}$$

Definition 15. *The $\mathcal{K}$-codimension of a germ $[f] \in \mathcal{Z}^p_n$, denoted by $\operatorname{codim}_{\mathcal{K}}([f])$, is the codimension in $\mathcal{E}^p_n$ of the linear subspace $T_{\mathcal{K},e}f$, or, which is the same,*

$$\operatorname{codim}_{\mathcal{K}}([f]) = \dim\left(\mathcal{E}^p_n / T_{\mathcal{K},e}f\right). \tag{51}$$

Having defined codimension in the context of $\mathcal{K}$-equivalence, we proceed to stating two fundamental results relating versality and codimension.

Proposition 4. *Two $\mathcal{K}$-equivalent germs have the same $\mathcal{K}$-codimension.*

Theorem 6. *Let $[f] \in \mathcal{Z}^p_n$ be such that $\operatorname{codim}_{\mathcal{K}}([f]) = d$. The following hold:*

(1) An unfolding $[\tilde{F}] \in \mathcal{Z}^p_{n,k}$ of $[f]$ with a representative of the form

$$\tilde{F}(x, \eta_1, \dots, \eta_k) = (F(x, \eta_1, \dots, \eta_k), \eta_1, \dots, \eta_k)$$

is $\mathcal{K}$-versal if, and only if,

$$T_{\mathcal{K},e}f + \operatorname{span}_{\mathbb{R}}\left(\left[\frac{\partial F}{\partial \eta_1}\Big|_{\eta=0}\right], \dots, \left[\frac{\partial F}{\partial \eta_k}\Big|_{\eta=0}\right]\right) = \mathcal{E}^p_n.$$

(2) There is $[\tilde{H}] \in \mathcal{Z}^p_{n,d}$ that is a $\mathcal{K}$-versal unfolding of $[f]$.

(3) If $[\tilde{F}], [\tilde{G}] \in \mathcal{Z}^p_{n,k}$ are $\mathcal{K}$-versal unfoldings of $[f]$, then they are $\mathcal{K}$-equivalent.

The concept of codimension is extremely important in the context of this paper because of its role defining the concept of $\mathcal{K}$-universality.

Definition 16. *Let $[f] \in \mathcal{Z}_n^p$ be such that* $\operatorname{codim}_{\mathcal{K}}([f]) = d$*. A $\mathcal{K}$-versal unfolding $[\tilde{F}] \in \mathcal{Z}_{n,k}^p$ of $[f]$ is said to be **$\mathcal{K}$-universal** if $k = d$, that is, the number of parameters of the unfolding $[\tilde{F}]$ is equal to the codimension of $[f]$.*

An important property of the codimension of a germ is established by Proposition 5. Proposition 6, on the other hand, characterises $\mathcal{K}$-universal unfoldings.

Proposition 5. *Let $[f] \in \mathcal{Z}_n^p$ be such that* $\operatorname{codim}_{\mathcal{K}}([f]) = d$*. Then, d is the minimal number of parameters that an unfolding of $[f]$ must have to be $\mathcal{K}$-versal.*

Proposition 6. *An unfolding $[\tilde{F}] \in \mathcal{Z}_{n+k}^p$ of $[f] \in \mathcal{Z}_{n^p}$ is $\mathcal{K}$-universal if, and only if,*

(1) $\left\{ \left[\frac{\partial F}{\partial \eta_1}\Big|_{\eta=0} \right], \dots, \left[\frac{\partial F}{\partial \eta_k}\Big|_{\eta=0} \right] \right\} \subset \mathcal{E}_n^p$ *is linearly independent;*

(2) $T_{\mathcal{K},e} f \oplus \operatorname{span}_{\mathbb{R}} \left(\left[\frac{\partial F}{\partial \eta_1}\Big|_{\eta=0} \right], \dots, \left[\frac{\partial F}{\partial \eta_k}\Big|_{\eta=0} \right] \right) = \mathcal{E}_n^p.$

We now define the pushforward of an unfolding, a concept that will be important for establishing the idea of equivalent families of two distinct but equivalent germs.

Definition 17. *Let $[\tilde{F}] \in \mathcal{Z}_{n,k}^p$ be an unfolding of the germ $[f] \in \mathcal{Z}_n^p$. For any given pair $([M],[\phi]) \in GL_p(\mathcal{E}_n) \times L_n$, the pushforward of $[\tilde{F}]$ by $([M],[\phi])$, denoted by $([M],[\phi]) * [\tilde{F}]$ is defined as the unfolding of the germ $[f_{push}] = [M] \cdot [f] \circ [\phi]$ whose representative $\tilde{F}_{push}(x,\eta) = (F_{push}(x,\eta),\eta)$ satisfies*

$$F_{push}(x,\eta) = M(x) \cdot F(\phi(x),\eta), \tag{52}$$

in a neighbourhood of the origin.

The pushforward has the important property of preserving induction, as proved in the next proposition.

Proposition 7. *Let $[f] \in \mathcal{Z}_n^p$ and $([M],[\phi]) \in GL_p(\mathcal{E}_n) \times L_n$. The pushforward $([M],[\phi])*$ is a bijective map between the set of unfoldings of $[f]$ and unfoldings of $[f_{push}] = [M] \cdot [f] \circ [\phi]$ that preserves $\mathcal{K}$-induction.*

Proof. It is bijective because it has an inverse given by $([M^{-1}],[\phi^{-1}])$.

Suppose that $[\tilde{G}] \in \mathcal{E}_{n,l}^p$ is $\mathcal{K}$-induced by $[\tilde{F}]$ via $[h]$. Then, there are neighbourhoods of the origin $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^k$, and smooth functions $\alpha : U \times V \to \mathbb{R}^n$ and $Q : U \times V \to \mathbb{R}^{p\times p}$ such that $h(0) = 0$, $\alpha(x,0) = x$, $Q(x,0) = I_p$, and

$$G(x,\eta) = Q(x,\eta) \cdot F(\alpha(x,\eta),h(\eta)) \tag{53}$$

for $(x,\eta) \in U \times V$.

Define $[\tilde{F}_{\text{push}}] = ([M],[\phi]) * [\tilde{F}]$ and $[\tilde{G}_{\text{push}}] = ([M],[\phi]) * [\tilde{G}]$. By definition, if $\tilde{F}_{\text{push}}(x,\eta) = (F_{\text{push}}(x,\eta),\eta)$ and $\tilde{G}_{\text{push}}(x,\eta) = (G_{\text{push}}(x,\eta),\eta)$ are representatives, then

$$G_{\text{push}}(x,\eta) = M(x) \cdot Q(\phi(x),\eta) \cdot F(\alpha(\phi(x),\eta),h(\eta)). \tag{54}$$

Setting $\beta(x,\eta) = \phi^{-1}(\alpha(\phi(x),\eta))$ and $S(x,\eta) = M(x)Q(\phi(x),\eta)M^{-1}(x)$, it follows that

$$G_{\text{push}}(x,\eta) = S(x,\eta) \cdot F_{\text{push}}(\beta(x,\eta),h(\eta)), \tag{55}$$

which proves that $[\tilde{G}_{\text{push}}]$ is $\mathcal{K}$-induced by $[\tilde{F}_{\text{push}}]$ via $[h]$. □

Corollary 2. *The pushforward of $\mathcal{K}$-equivalent unfoldings are $\mathcal{K}$-equivalent. Also, the pushforward of a $\mathcal{K}$-versal unfolding is $\mathcal{K}$-versal.*

4.3. **Determining versality and codimension.** This section is dedicated to providing a framework of a classification of bifurcation families which should help in answering the question of whether that family is a versal unfolding of its singular germ.

The main concepts allowing for a simplification of the problem are that of the corank and the core of a singularity. The introduction of those concepts ensure that, in multidimensional maps, we can safely ignore "non-singular" dimensions of a singularity, taking into account only its "singular part".

This potentially simplifies the comprehension of apparently complicated families with many parameters near a singularity, as it can eliminate many of those parameters, retaining only the ones essential in guaranteeing a sufficiently general unfolding.

The content of this section is not new, except in the way its formulated. Equivalent results can be found in [33], for instance.

4.3.1. *Corank and core of a singularity.* It has been established that, if $x \mapsto F(x,\mu,\varepsilon)$ is a family of maps that is $\mathcal{K}$-equivalent to $x \mapsto \Delta_\ell(x,\mu,\varepsilon)$, the zero sets $F^{-1}(0)$ and $\Delta_\ell^{-1}(0)$ can be transformed into each other via a fibred diffeomorphism. This will allow us to describe the zero set of Δ_ℓ by finding an adequate F, which is in the simplest form possible, in each of the cases we study.

In assuming items (H1) and (H2), we are saying that $0 \in \mathbb{R}^n$ a singular zero of the map $x \mapsto g_\ell(x,0) = \Delta_\ell(x,0,0)$. The first step in finding the adequate F described in the previous paragraph is determining the so-called corank of this singularity, defined as follows:

Definition 18. *Let $U \subset \mathbb{R}^n$ be open, $p \in U$, and $f : U \to \mathbb{R}^m$. The point p is said to be of corank c if the corank of the derivative $f'(p)$ is c, that is,*

$$c = n - \operatorname{rank} f'(p). \tag{56}$$

Any point whose corank is at least 1 is called a singularity of the map f.

Before proceeding, we would like to briefly remark that items (H1) and (H2) imply not only that 0 is a singularity of the map $x \mapsto g_\ell(x,0)$, but also that it is an equilibrium of the vector field given by this map.

Knowing the corank of zero for the map $x \mapsto g_\ell(x,0)$ can greatly simplify the problem of determining versality and codimension of the family $g_\ell(x,\mu)$, due to the following result.

Proposition 8. *Let $[\tilde{F}] \in \mathcal{Z}^n_{n,k}$ be a k-parameter unfolding of $[f] \in \mathcal{Z}^n_n$. Suppose that f has a corank $c \leq p$ singularity at 0. There are $[g] \in \mathcal{Z}^n_n$ and an unfolding $[\tilde{G}] \in \mathcal{Z}^n_{n,k}$ of $[g]$ such that*

- *(a) $[g]$ and $[f]$ are $\mathcal{K}$-equivalent and a representative g satisfies $g(y,z) = (g_c(y), z) \in \mathbb{R}^c \times \mathbb{R}^{n-c}$;*
- *(b) Let $([M],[\phi]) \in GL_n(\mathcal{E}_n) \times L_n$ be such that $[g] = [M] \cdot [f] \circ [\phi]$. The pushforward $([M],[\phi]) * [\tilde{F}]$ is $\mathcal{K}$-equivalent to $[\tilde{G}]$ via the identity and a representative $\tilde{G}(y,z,\eta) = (G(y,z,\eta),\eta)$ satisfies $G(y,z,\eta) = (G_c(y,\eta), z) \in \mathbb{R}^c \times \mathbb{R}^{n-c}$.*

Proof. Since $f = (f_1,\ldots,f_n)$ has corank c, it follows that we can multiply f on the left by an invertible matrix $E \in GL_n(\mathbb{R})$ to change the order of the coordinate functions f_i so that $E \cdot f = (f_{i_1},\ldots,f_{i_n})$ is such that the set of line-matrices $\{f'_{i_{c+1}}(0),\ldots,f'_{i_n}(0)\}$ is linearly independent.

Let $\tilde{F}(x,\eta) = (F(x,\eta),\eta) = (F_1(x,\eta),\ldots,F_n(x,\eta),\eta)$ and define the functions p_a and p_b by $p_a(x,\eta) = (F_{i_1}(x,\eta),\ldots,F_{i_c}(x,\eta))$ and $p_b(x,\eta) = (F_{i_{c+1}}(x,\eta),\ldots,F_{i_n}(x,\eta))$. This means that, defining $\tilde{E} = E \oplus I_k$, we have

$$\tilde{E} \cdot \tilde{F}(x,\eta) = (p_a(x,\eta), p_b(x,\eta), \eta) \in \mathbb{R}^c \times \mathbb{R}^{n-c} \times \mathbb{R}^k. \tag{57}$$

If $\pi_\eta(x,\eta) = \eta$ is the standard projection, considering that (p_b, π_η) is a submersion near $(x,\eta) = (0,0)$, the Local Submersion Theorem ensures that there is a diffeomorphism $\tilde{\phi}(x,\eta)$ such that $\tilde{\phi}(0,0) = (0,0)$, $p_b \circ \tilde{\phi}(x,\eta) = (x_{c+1},\dots,x_n) = z$ and $\pi_\eta \circ \tilde{\phi}(x,\eta) = \eta$. From the third identity, in particular, it follows that $\tilde{\phi}(x,\eta) = (\tilde{\phi}_c(x,\eta),\eta)$.

Henceforth, we adopt the notation $y = (x_1,\dots,x_c)$ and $z = (x_{c+1},\dots,x_n)$. Observe that Hadamard's Lemma yields

$$(p_a \circ \tilde{\phi})(y,z,\eta) - (p_a \circ \tilde{\phi})(y,0,\eta) = R(y,z,\eta)\cdot z \tag{58}$$

which can be rewritten as

$$\tilde{E}\cdot\tilde{F}\circ\tilde{\phi}(y,z,\eta) = \begin{bmatrix} I_c & R(y,z,\eta) & 0 \\ 0 & I_{n-c} & 0 \\ 0 & 0 & I_k \end{bmatrix}\begin{bmatrix} p_a\circ\tilde{\phi}(y,0,\eta) \\ z \\ \eta \end{bmatrix}. \tag{59}$$

Define $G_c(y,\eta) := \tilde{p}_a \circ \tilde{\phi}(y,0,\eta)$, $G(y,z,\eta) = (G_c(y,\eta),z)$, and $\tilde{G}(y,z,\eta) = (G(y,z,\eta),\eta)$. It is clear that $[\tilde{G}] \in \mathcal{Z}^n_{n,k}$ is an unfolding of the germ of $g : (y,z) \mapsto G(y,z,0)$, which has the desired form with $g_c(y) = G_c(y,0)$. Moreover, since $\tilde{E} = E \oplus I_k$ and

$$\begin{bmatrix} I_c & \tilde{R}(y,z,\eta) & 0 \\ 0 & I_{n-c} & 0 \\ 0 & 0 & I_k \end{bmatrix}^{-1} = \begin{bmatrix} I_c & -\tilde{R}(y,z,\eta) & 0 \\ 0 & I_{n-c} & 0 \\ 0 & 0 & I_k \end{bmatrix}, \tag{60}$$

by defining

$$S(y,z,\eta) = \begin{bmatrix} I_c & -\tilde{R}(y,z,\eta) \\ 0 & I_{n-c} \end{bmatrix} E, \tag{61}$$

and considering (59), it follows that $G(y,z,\eta) = S(y,z,\eta)\cdot F\circ\tilde{\phi}_c(y,z,\eta)$.

Therefore, setting $M(y,z) := S(y,z,0)$ and $\phi(y,z) := \tilde{\phi}_c(y,z,0)$, it follows, on the one hand, that $[g] = [M]\cdot[f]\circ[\phi]$, proving item (a). On the other hand, item (b) follows by defining the matrix function $Q(y,z,\eta) = S(y,z,\eta)(M(y,z))^{-1}$, the function $\alpha(y,z,\eta) := \phi^{-1}\circ\tilde{\phi}_c(y,z,\eta)$ and verifying that they satisfy the conditions present in Definition 12 with $h(\eta) = \eta$. □

Inspired by that result, we define the following concepts.

Definition 19. *Let $[\tilde{F}] \in \mathcal{Z}^n_{n,k}$ be a k-parameter unfolding of $[f] \in \mathcal{Z}^n_n$. Suppose that f has a corank $c \le p$ singularity at 0. Any germ $[g_c] \in \mathcal{Z}^c_c$ as given in Proposition 8 is called a core of the germ $[f]$. Similarly, any $[G_c]$ as given in that Proposition is a core of the germ of the family $[F]$.*

Essentially, Proposition 8 allows us to conclude that the zero set of the family $\Delta_\ell(x,\mu,\varepsilon)$ near a singularity of corank c is fibred-diffeomorphic to the zero set of a family of the form $(y,z,\mu,\varepsilon)\mapsto(\mathcal{X}_\ell(y,\mu,\varepsilon),z)\in\mathbb{R}^c\times\mathbb{R}^{n-c}$. Hence, when considering singularities of corank c appearing in $x\mapsto\Delta_\ell(x,0,0)$, we are justified in assuming, for convenience, that $n = c$, because the zero set of Δ_ℓ will be equivalent to the points (y,μ,ε) satisfying $\mathcal{X}_\ell(y,\mu,\varepsilon) = 0$. We remark that the notation $\mathcal{X}_\ell$ will be reserved to the core of Δ_ℓ.

Two crucial observations regarding $\mathcal{X}_\ell$ should be kept in mind. First, that it captures all the behaviour concerning bifurcation of zeroes of Δ_ℓ - thus also the behaviour concerning bifurcation of fixed points of Π. Secondly, since $\Delta_\ell(x,\mu,0) = g_\ell(x,\mu)$, the map

$(x,\mu) \mapsto \mathcal{X}_\ell(x,\mu,0)$, obtained by fixing $\varepsilon = 0$, is given by the core of the family $g_\ell(x,\mu)$ itself. Combining those two observations, we conclude that the bifurcation of equilibria of the guiding system $\dot{x} = g_\ell(x,\mu)$ is exactly reproduced in the zeroes of $(x,\mu) \mapsto \mathcal{X}_\ell(x,\mu,0)$.

4.4. **Stable families with corank 1 singularities.** In this section, we introduce a classification of $\mathcal{K}$-universal unfoldings of corank 1 singularities based on Thom's elementary catastrophes.

By Proposition 8, any $\mathcal{K}$-universal unfolding $[\tilde{F}] \in \mathcal{Z}^n_{n,k}$ of a corank 1 singularity $[f] \in \mathcal{Z}^n_n$ corresponds to a singular germ $[g] \in \mathcal{Z}^n_n$ of the form $g(y,z) = (g_c(y), z) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and an unfolding $[\tilde{G}]$ of $[g]$ satisfying $\tilde{G}(y,z,\eta) = (G_c(y,\eta), z, \eta)$. The classification of those unfoldings can then be achieved by taking into account the classification of the simpler germs $[g_c] \in \mathcal{Z}^1_1$ and $[G_c] \in \mathcal{Z}^1_{1,k}$ that is provided by Thom's catastrophe theory, with a caveat: in many treatments of Thom's theory, since it is often assumed that potential functions are the object, a slightly different equivalence relation is utilized (see, for instance, the concept of $\mathcal{R}_+$-equivalence in [33]). We remark, however, that for $n = 1$ this assumption is inconsequential.

More importantly, since the pushforward of an unfolding preserves versality (and universality), one obtains a versal unfolding of $[f]$ if a versal unfolding of $[g_c]$ is given, and vice-versa. Therefore, the classification put forth above is a complete description of $\mathcal{K}$-universal unfoldings of corank 1 singularities.

For germs of lower codimension, we even take the liberty of adopting the usual names of catastrophes appearing in Thom's theory, as follows.

Definition 20. *A k-parameter family of n-dimensional vector fields $F(x,\mu)$ is said to undergo a*

- ***fold*** *catastrophe, if $k = 1$;*
- ***cusp*** *catastrophe, if $k = 2$;*
- ***swallowtail*** *catastrophe, if $k = 3$;*
- ***butterfly*** *catastrophe, if $k = 4$;*
- ***wigwam*** *catastrophe, if $k = 5$;*
- ***star*** *catastrophe, if $k = 6$;*

at the origin for $\mu = 0$ if it satisfies:

1. *The germ of $f : x \mapsto F(x,0)$ at the origin is $\mathcal{K}$-equivalent to the germ of $s_{1^k,0}(y,z) = (y^{k+1}, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$;*
2. *Let $([M],[\phi]) \in GL_n(\mathcal{E}_n) \times L_n$ be such that $[f] = [M] \cdot [s_{1^k,0}] \circ [\phi]$. The pushforward $([M],[\phi]) * [\tilde{\mathcal{U}}]$ of the unfolding $[\tilde{\mathcal{U}}] \in \mathcal{Z}^n_{n,k}$, given by $\tilde{\mathcal{U}}(y,z,\eta) = (\mathcal{U}(y,\eta), z, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^k$, $\eta = (\eta_1, \ldots, \eta_k)$, and*

$$\mathcal{U}(y,\eta) = y^{k+1} + \eta_1 y^{k-1} + \eta_2 y^{k-2} + \ldots \eta_k \tag{62}$$

   is $\mathcal{K}$-equivalent to $[\tilde{F}]$ via the identity, where $\tilde{F}(x,\mu) = (F(x,\mu),\mu)$;

This definition only respects the number of equilibria of $F(x,\mu)$, not the topological equivalence class of F, and so in [20] these were called *underlying catastrophes* of the vector field.

4.5. **The averaging method: a brief presentation.** As mentioned in Section 2.2, the averaging method is the tool that allows us to transform the system we are interested in studying into a form that is amenable to analysis. In this section, we briefly introduce this method and provide explicit formulas for calculating the terms appearing in the definition of Δ_ℓ, the displacement function of order ℓ.

4.5.1. *Main concepts and historical development.* A widely used technique for analysing nonlinear oscillatory systems under small perturbations is the averaging method. This method has been rigorously formalised in a series of works starting with Fatou [14] and including those by Krylov and Bogoliubov [24], Bogoliubov [5], and later by Bogoliubov and Mitropolsky [4]. However, its origins can be traced back to the early perturbative methods applied in the study of solar system dynamics by Clairaut, Laplace, and Lagrange. For a concise historical overview of averaging theory, see [32, Chapter 6] and [43, Appendix A].

A key result from the averaging theory (see [43, Lemma 2.9.1]) states the existence of a smooth near-identity map

$$X = U(t, z, \mu, \varepsilon) = z + \sum_{i=1}^{k} \varepsilon^i u_i(t, z, \mu), \tag{63}$$

T-periodic in t, that transforms differential equation (1) into:

$$z' = \sum_{i=1}^{k} \varepsilon^i g_i(z, \mu) + \varepsilon^{k+1} r_k(t, z, \mu, \varepsilon). \tag{64}$$

By imposing the existence of such a transformation and, then, solving homological equations, the functions u_i and g_i, for $i \in \{1, 2, \dots, k\}$, can be recursively obtained. In general,

$$g_1(z, \mu) = \frac{1}{T} \int_0^T F_1(t, z, \mu) dt$$

but, for $i \in \{2, \dots, k\}$, such functions are not unique. Nevertheless, by imposing additionally the *stroboscopic condition*

$$U(z, 0, \mu, \varepsilon) = z,$$

each g_i becomes uniquely determined, which is referred to as the *stroboscopic averaged function of order i* (or simply *ith-order averaged function*) of (1).

The primary objective of the averaging method consists in estimating the solutions of the non-autonomous original differential equation (1) by means of the following autonomous differential equation

$$z' = \sum_{i=1}^{k} \varepsilon^i g_i(z, \mu), \tag{65}$$

which corresponds to the truncation up to order k in ε of the differential equation (64). Accordingly, for sufficiently small $|\varepsilon| \neq 0$, the solutions of (1) and (65), with identical initial condition, remain ε^k-close over an interval of time of size $\mathcal{O}(1/\varepsilon)$ (see [43, Theorem 2.9.2]).

The averaging method has shown to be highly effective in detecting the emergence of invariant structures of (1) that originate from hyperbolic invariant structures of the autonomous differential equation

$$z' = g_\ell(z), \tag{66}$$

where g_ℓ is the first averaged function that is not identically zero. As we have mentioned in Section 2, (66) is known as *guiding system*. A classical result of averaging theory in this context asserts the birth of an isolated periodic solution of (1) provided that the guiding system (66) has a simple equilibrium (see, for instance, [17, 45]). This result has been extended to settings with less regularity [2, 6, 26, 27, 29, 36, 39]. More recently, in [38, 40], this result has been generalised to detect higher-dimensional structures. Specifically, it has been proven that the differential equation (1) possesses an invariant torus, provided there is a hyperbolic limit cycle in the the guiding system (66) (see also [12]).

4.5.2. *Calculation of the averaged functions.* As recently highlighted in [35], the averaging method is strictly related with the Melnikov method, which consists in expanding the solutions $X(t, X_0, \mu, \varepsilon)$ of (1), satisfying $X(0, X_0, \mu, \varepsilon) = X_0$, around $\varepsilon = 0$ as (see [28, 34]):

$$X(t, X_0, \mu, \varepsilon) = X_0 + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t, X_0, \mu)}{i!} + \varepsilon^{k+1} r_k(t, X_0, \mu, \varepsilon), \tag{67}$$

where

(68)

$$y_1(t, X_0, \mu) = \int_0^t F_1(s, X_0, \mu)\, ds \quad \text{and}$$

$$y_i(t, X_0, \mu) = \int_0^t \left(i! F_i(s, X_0, \mu) + \sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} \partial_x^m F_{i-j}(s, X_0, \mu) B_{j,m}\big(y_1, \ldots, y_{j-m+1}\big)(s, X_0, \mu) \right) ds,$$

for $i \in \{2, \ldots, k\}$. Here, $B_{p,q}$ refers to the *partial Bell polynomials* [10]. We emphasise that the formula presented above can be easily implemented in algebraic softwares such as Mathematica and Maple.

From (67), the stroboscopic Poincaré map writes

$$\Pi(X_0, \mu, \varepsilon) = X(T, X_0, \mu, \varepsilon) = X_0 + \sum_{i=1}^{k} \varepsilon^i f_i(X_0, \mu) + \varepsilon^{k+1} R_k(X_0, \mu, \varepsilon)$$

where $R_k(X_0, \mu, \varepsilon) = r_k(T, X_0, \mu, \varepsilon)$ and, for each i,

$$f_i(X_0, \mu) = \frac{y_i(T, X_0, \mu)}{i!} \tag{69}$$

Notice that $f_1 = T g_1$. The function f_i is referred to as the *Poincaré-Pontryagin-Melnikov function of order i* or simply *ith-order Melnikov function*. It should be noted that, in the literature, these functions are sometimes also called *averaged functions* (see, for instance, [28]).

The next formula connecting averaged and Melnikov functions was established in [35, Theorem A]:

$$\begin{aligned} g_1(z, \mu) &= \frac{1}{T} f_1(z, \mu), \\ g_i(z, \mu) &= \frac{1}{T} \left(f_i(z, \mu) - \sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{1}{j!} d^m g_{i-j}(z, \mu) \int_0^T B_{j,m}\big(\tilde{y}_1, \ldots, \tilde{y}_{j-m+1}\big)(s, z, \mu) ds \right), \end{aligned} \tag{70}$$

with $\tilde{y}_i(t, z, \mu)$, for $i \in \{1, \ldots, k\}$, being recursively defined by:

$$\begin{aligned} \tilde{y}_1(t, z, \mu) &= t\, g_1(z, \mu) \\ \tilde{y}_i(t, z, \mu) &= i! t\, g_i(z, \mu) + \sum_{j=1}^{i-1} \sum_{m=1}^{j} \frac{i!}{j!} d^m g_{i-j}(z, \mu) \int_0^t B_{j,m}\big(\tilde{y}_1, \ldots, \tilde{y}_{j-m+1}\big)(s, z, \mu) ds. \end{aligned} \tag{71}$$

The preceding formula facilitates the calculation of the averaged functions without the need of handling the near-identity transformation (63) and solving homological equations. For a practical implementation of this formula, we refer to [35, Appendix A], where a Mathematica algorithm is provided for computing the averaging functions.

As previously emphasised, the guiding system (66) plays a crucial role in averaging theory and is defined by the first averaged function that is not identically zero. In the following proposition, it is provided a straightforward and easily computable formula for this function, as well as for some of the subsequent averaged functions, using Melnikov functions:

Proposition 9. [37, Proposition 1] *Let $\ell \in \{2, \dots, k\}$. If either $f_1 = \cdots = f_{\ell-1} = 0$ or $g_1 = \cdots = g_{\ell-1} = 0$, then*

$$g_i = \frac{1}{T} f_i, \quad \text{for} \quad i \in \{1, \dots, 2\ell - 1\},$$

and

$$g_{2\ell}(z, \mu) = \frac{1}{T} \left(f_{2\ell}(z, \mu) - \frac{1}{2} d f_\ell(z, \mu) \cdot f_\ell(z, \mu) \right).$$

5. Proof of theorems

This section contains the proofs for all the theorems stated in Section 2, as well as some novel auxiliary results that are used in those proofs.

5.1. Auxiliary results: zero sets and $\mathcal{K}$-equivalence.

We first present and prove some useful (albeit slightly technical) results concerning the set of zeroes of different unfoldings of the same germ. The general idea of $\mathcal{K}$-equivalence preserving the zero sets of germs, and thus being useful in the study of bifurcations, is well-known (see [30, 33]). Here, we prove a precise formulation of that idea suited to our context. In particular, our results allow us to more easily consider strongly-fibred diffeomorphisms, maintaining the difference between the "bifurcation" parameters μ and the perturbation parameter ε.

Lemma 1. *Let $[\tilde{F}], [\tilde{G}] \in \mathcal{Z}^p_{n,k}$ be unfoldings of, respectively, $[f], [g] \in \mathcal{Z}^p_n$. Assume that $[f]$ and $[g]$ are $\mathcal{K}$-equivalent and let $([M], [\phi]) \in GL_p(\mathcal{E}_n) \times L_n$ be such that $[g] = [M] \cdot [f] \circ [\phi]$. Also, let $\tilde{F} : \mathcal{D} \times \Sigma_k \to \mathbb{R}^p \times \mathbb{R}^k$, $\tilde{G} : \mathcal{D} \times \Sigma_k \to \mathbb{R}^p \times \mathbb{R}^k$ be representatives of $[\tilde{F}]$ and $[\tilde{G}]$ of the form $\tilde{F}(x, \eta) = (F(x, \eta), \eta)$ and $\tilde{G}(x, \eta) = (G(x, \eta), \eta)$.*

If $[\tilde{G}]$ is $\mathcal{K}$-equivalent to $([M], [\phi]) * [\tilde{F}]$ via $[h]$, there are $W \in \mathcal{N}_0(\mathcal{D} \times \Sigma)$ and a diffeomorphism $\Phi : W \to E_\Phi$, satisfying $\Phi(x, \eta) = (\Phi_1(x, \eta), \Phi_2(\eta)) \in \mathbb{R}^n \times \mathbb{R}^k$, $\Phi(x, 0) = (\phi(x), 0)$, and

$$Z_F \cap E_\Phi = \Phi\left(Z_G \cap W\right). \tag{72}$$

Additionally, if F is independent of the last $k_F \in \{0, 1, \dots, k-1\}$ entries of $\eta = (\eta_1, \dots, \eta_k)$ and $h = (h_1, \dots, h_k)$ is such that

$$\det \left[\frac{\partial \left(h_1, \dots, h_{k-k_F}\right)}{\partial(\eta_1, \dots, \eta_{k-k_F})}(0, 0) \right] \neq 0; \tag{73}$$

then Φ_2 can be chosen as $\Phi_2(\eta) = (h_1(\eta), \dots, h_{k-k_F}, \eta_{k-k_F+1}, \dots, \eta_k)$. In particular, $\Phi_2 = h$ can be chosen regardless of F.

Proof. Since $[\tilde{G}]$ is $\mathcal{K}$-equivalent to $([M], [\phi]) * [\tilde{F}]$ via the local diffeomorphism germ $[h]$, then a representative $h : \Sigma'_k \to \Sigma'_k$ is such that $h(0) = 0$ and $h'(0)$ is invertible. We can assume that $\Sigma'_k \in \mathcal{N}_0(\Sigma_k)$ is sufficiently small to ensure that $h'(\eta)$ is invertible on Σ'_k. Moreover, there are $U_0 \in \mathcal{N}_0(\mathcal{D})$, $V_0 \in \mathcal{N}_0(\Sigma'_k)$, and smooth functions $Q(x, \eta)$ and $\alpha(x, \eta)$ such that $Q(x, 0) = I_p$, $\alpha(x, 0) = x$, and

$$G(x, \eta) = Q(x, \eta) M(\alpha(x, \eta)) F(\phi(\alpha(x, \eta)), h(\eta)) \tag{74}$$

for any $(x, \eta) \in U_0 \times V_0$. Without loss of generality, we assume that $\overline{U}_0 \subset \mathcal{D}$

Since α and Q are smooth, $\alpha(x,0) = x$, and $Q(x,0) = I_p$, we can find $U_1 \in \mathcal{N}_0(U_0)$ and $V_1 \in \mathcal{N}_0(V_0)$ sufficiently small as to guarantee that $\alpha'_\eta(x)$ and $Q(x,\eta)$ are invertible for $(x,\eta) \in U_1 \times V_1$ and that $\alpha(U_1 \times V_1)$ is in contained in a set where M and ϕ are invertible.

Define $W = U_1 \times V_1$ and $\Phi(x,\eta) = (\phi(\alpha(x,\eta)), h(\eta))$, which is clearly of the desired form. Since

$$\det \Phi'(x,\eta) = \det \phi'(\alpha(x,\eta)) \cdot \det \alpha'_\eta(x) \cdot \det h'(\eta), \tag{75}$$

it follows that $\Phi'(x,\eta)$ is invertible for $(x,\eta) \in W$. Hence, Φ is a diffeomorphism on W, and it is easy to see that $\Phi(x,0) = (\phi(x),0)$. Let E_Φ be the image $\Phi(W)$.

For the relationship between the Z_F and Z_G, observe that, on the one hand, if $(x,\eta) \in Z_G \cap W$, then, by (74),

$$F(\Phi(x,\eta)) = F(\phi(\alpha(x,\eta)), h(\eta)) = (M(\alpha(x,\eta)))^{-1} \, (Q(x,\eta))^{-1} \, G(x,\eta) = 0, \tag{76}$$

so that $\Phi(x,\eta) \in Z_G \cap E_\Phi$. On the other hand, if $(y,\xi) \in Z_F \cap E_\Phi$, (74) ensures that $(x,\eta) := \Phi^{-1}(y,\xi)$ satisfies

$$G(x,\eta) = Q(x,\eta)M(\alpha(x,\eta))F(\Phi(x,\eta)) = Q(x,\eta)G(y,\xi) = 0, \tag{77}$$

so that $\Phi^{-1}(y,\xi) \in Z_G \cap W$. This proves (72).

Suppose now that the additional hypotheses in the statement of the theorem hold, that is, F is independent of the last $k_F < k$ entries of η and (73) is valid. Define

$$h_{\mathrm{Fib}}(\eta) = \left(h_1(\eta), \ldots, h_{k-k_F}(\eta), \eta_{k-k_F+1}, \ldots, \eta_k\right), \tag{78}$$

It is easy to see that (73) ensures h_{Fib} is a local diffeomorphism near the origin. Moreover, the independence of F with respect to its last k_F entries guarantees that (74) still holds after replacing h with h_{Fib}.

By retracing the steps of the proof with this new h_{Fib}, we obtain the analogous of (72) with Φ replaced by $\Phi_{\mathrm{Fib}}(x,\eta) = (\phi(\alpha(x,\eta)), h_{\mathrm{Fib}}(\eta))$, which is clearly of the desired form. □

Remark 1. *Two unfoldings $[\tilde{F}], [\tilde{G}] \in \mathcal{Z}^p_{n,k}$ of $\mathcal{K}$-equivalent germs satisfying the hypotheses of Lemma 1 are said to be $\mathcal{K}$-equivalent as families, even though it should be kept in mind that they are not unfoldings of the same germ, and thus cannot be considered equivalent unfoldings.*

The next result is used to connect our hypothesis of $\mathcal{K}$-universality to the hypotheses of Lemma 1. It is an important technical step in proving the main theorem of this paper.

Lemma 2. *Let $[f] \in \mathcal{Z}^p_n$ a germ of $\mathcal{K}$-codimension d, and $[\tilde{H}] \in \mathcal{Z}^p_{n,d}$ be a $\mathcal{K}$-universal unfolding of $[f]$. Also, let $k \geq 0$ and $[\tilde{F}] \in \mathcal{Z}^p_{n,d+k}$ be an unfolding of $[f]$. Take $\tilde{H} : \mathcal{D} \times \Sigma_d \to \mathbb{R}^p \times \mathbb{R}^d$ and $\tilde{F} : \mathcal{D} \times \Sigma_{d+k} \to \mathbb{R}^p \times \mathbb{R}^{d+k}$ to be representatives of the form $\tilde{H}(x,\eta) = (H(x,\eta),\eta)$, $\tilde{F}(x,\eta,\xi) = (F(x,\eta,\mu),\eta,\xi)$, and assume that $F(x,\eta,0) = H(x,\eta)$.*

Suppose that $[\tilde{F}]$ is $\mathcal{K}$-induced by $[\tilde{H}]$ via $[h] : (\mathbb{R}^{d+k},0) \to (\mathbb{R}^d,0)$. Then,

$$\det\left(\frac{\partial h}{\partial \eta}(0,0)\right) \neq 0. \tag{79}$$

Proof. Since $[\tilde{F}]$ is $\mathcal{K}$-induced by $[\tilde{H}]$ via $[h]$, it follows that there are $U_0 \in \mathcal{N}_0(\mathcal{D})$, $V_0 \in \mathcal{N}_0(\Sigma_{d+k})$, and smooth functions $Q(x,\eta,\xi)$ and $\alpha(x,\eta,\xi)$ such that $Q(x,0,0) = I_p$, $\alpha(x,0,0) = x$, and

$$F(x,\eta,\xi) = Q(x,\eta,\xi) \cdot H(\alpha(x,\eta,\xi), h(\eta,\xi)). \tag{80}$$

By hypothesis, $F(x,\eta,0) = H(x,\eta)$, so that we obtain from (80) that

$$H(x,\eta) = Q(x,\eta,0) \cdot H(\alpha(x,\eta,0), h(\eta,0)). \tag{81}$$

For each x, we have an identity of smooth functions of η, which can thus be differentiated at $\eta = 0$ in the direction of $w \in \mathbb{R}^d$, yielding

(82)
$$\frac{\partial H}{\partial \eta}(x,0) \cdot w = \left(\frac{\partial Q}{\partial \eta}(x,0) \cdot w\right) \cdot H(x,0) + \frac{\partial H}{\partial x}(x,0) \cdot \frac{\partial \alpha}{\partial \eta}(x,0,0) \cdot w + \frac{\partial H}{\partial \eta}(x,0) \cdot \frac{\partial h}{\partial \eta}(0,0) \cdot w.$$

Let $w \in \mathbb{R}^d$ be given such that

$$\frac{\partial h}{\partial \eta}(0,0) \cdot w = 0. \tag{83}$$

We will show that $w = 0$, so that the derivative of h with respect to η at $(0,0)$ must be invertible. From (82), it follows that

$$\frac{\partial H}{\partial \eta}(x,0) \cdot w = \left(\frac{\partial Q}{\partial \eta}(x,0) \cdot w\right) \cdot H(x,0) + \frac{\partial H}{\partial x}(x,0) \cdot \frac{\partial \alpha}{\partial \eta}(x,0,0) \cdot w \tag{84}$$

Observe that, since $H(x,0) = f(x)$, the right-hand side of (84) is an element of the extended $\mathcal{K}$-tangent space $T_{\mathcal{K},e}f$. Moreover, it is clear that the left-hand side of the same identity belongs to the subspace

$$\operatorname{span}_{\mathbb{R}}\left(\left[\frac{\partial H}{\partial \eta_1}\Big|_{\eta=0}\right], \ldots, \left[\frac{\partial H}{\partial \eta_d}\Big|_{\eta=0}\right]\right). \tag{85}$$

Considering that, by hypothesis, $[\tilde{H}]$ is a $\mathcal{K}$-universal unfolding of f, we know by Proposition 6 that

$$T_{\mathcal{K},e}f \cap \operatorname{span}_{\mathbb{R}}\left(\left[\frac{\partial H}{\partial \eta_1}\Big|_{\eta=0}\right], \ldots, \left[\frac{\partial H}{\partial \eta_d}\Big|_{\eta=0}\right]\right) = \{0\}. \tag{86}$$

Thus, it follows at once that

$$\frac{\partial H}{\partial \eta}(x,0) \cdot w = 0. \tag{87}$$

If $w \neq 0$, then there would be a non-trivial linear combination of elements of

$$\left\{\left[\frac{\partial H}{\partial \eta_1}\Big|_{\eta=0}\right], \ldots, \left[\frac{\partial H}{\partial \eta_d}\Big|_{\eta=0}\right]\right\} \tag{88}$$

that vanishes, contradicting the linear independence of this family established in Proposition 6.

Therefore, it follows that $w = 0$, concluding the proof. □

5.2. **Proof of the main result.** Having proved the auxiliary results above, we can proceed to proving the main result of this paper, Theorem 1. This is the purpose of this section and the proof is as follows.

By hypothesis, the germ $[\tilde{H}] \in \mathcal{Z}^n_{n,k}$ given by $\tilde{H}(x,\mu) = (H(x,\mu),\mu) = (g_\ell(x,\mu),\mu)$ is a $\mathcal{K}$-universal unfolding of the the germ $[s] \in \mathcal{Z}^n_n$ given by $s(x) = g_\ell(x,0)$. In particular, the unfolding $[\tilde{F}] \in \mathcal{Z}^n_{n,k+1}$ defined by $\tilde{F}(x,\mu,\varepsilon) = (F(x,\mu,\varepsilon),\mu,\varepsilon) = (\Delta(x,\mu,\varepsilon),\mu,\varepsilon)$ is $\mathcal{K}$-induced by $[\tilde{H}]$. Hence, let $Q(x,\mu,\varepsilon)$, $\alpha(x,\mu,\varepsilon)$ and $h(\mu,\varepsilon)$ be such that

$$F(x,\mu,\varepsilon) = Q(x,\mu,\varepsilon) \cdot H(\alpha(x,\mu,\varepsilon), h(\mu,\varepsilon)) \tag{89}$$

It is easy to see that, since $\Delta_\ell(x,\mu,0) = g_\ell(x,\mu)$, it follows that $F(x,\mu,0) = H(x,\mu)$. Thus, all the hypotheses of Lemma 2 are valid, ensuring that

$$\det\left(\frac{\partial h}{\partial \mu}(0,0)\right) \neq 0. \tag{90}$$

Define $[\tilde{G}] \in \mathcal{Z}^n_{n,k+1}$ by $\tilde{G}(x,\mu,\varepsilon) = (G(x,\mu,\varepsilon),\mu,\varepsilon) = (g_\ell(x,\mu),\mu,\varepsilon)$. In particular, we have that $G(x,\mu,\varepsilon) = H(x,\mu)$. Hence, (89) ensures that

$$F(x,\mu,\varepsilon) = Q(x,\mu,\varepsilon)\cdot G(\alpha(x,\mu,\varepsilon), h_{\text{ex}}(\mu,\varepsilon)), \tag{91}$$

where $h_{\text{ex}}(\mu,\varepsilon) = (h(\mu,\varepsilon),\varepsilon)$, which is clearly a local diffeomorphism near the origin of $\mathbb{R}^{k+1}$. Therefore, $[\tilde{F}]$ is $\mathcal{K}$-equivalent to $[\tilde{G}]$ via $[h_{\text{ex}}]$.

Finally, an application of Lemma 1 guarantees the existence of a diffeomorphism $\Phi : U \to V$, satisfying $\Phi(x,\mu,\varepsilon) = (\Phi_1(x,\mu,\varepsilon),\Phi_2(\mu,\varepsilon),\varepsilon) \in \mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}$, $\Phi(x,0,0) = (x,0,0)$, and

$$Z_F \cap V = \Phi\left(Z_G \cap U\right). \tag{92}$$

By definition of G, it is clear that $Z_G\cap U = (Z_{g_\ell}\times\mathbb{R})\cap U$. Similarly, $Z_F\cap V = Z_{\Delta_\ell}\cap V$. Thus, considering Corollary 1, it follows that $M_\Pi \cap V = \Phi\left((Z_{g_\ell}\times\mathbb{R})\cap U\right)\cup V_{\varepsilon=0}$. The fact that $Z_{g_\ell}\times\{0\}$ is invariant under Φ, on the other hand, follows from intersecting both sides of (92) with the set $\{(x,\mu,0)\in\mathbb{R}^3\}$, because the last coordinate function of Φ is ε identically. In fact, by doing so, we obtain

$$\left(Z_{g_\ell}\times\{0\}\right)\cap V = \Phi\left(\left(Z_{g_\ell}\times\{0\}\right)\cap U\right), \tag{93}$$

proving the invariance.

5.3. **Persistence of bifurcation diagrams: proof of Theorem 2.** In this section, we make use of Theorem 1 to prove the Theorem 2, concerning the persistence of bifurcation diagrams of equilibria.

Observe that $\mathcal{D}_{\ell,0}$ is defined by $\Delta_\ell(x,\mu,0) = 0$ and $\mathcal{D}_\varepsilon$ by $\Delta_\ell(x,\mu,\varepsilon) = 0$. The fact that $g_\ell(x,\mu) = \Delta_\ell(x,\mu,0)$ is $\mathcal{K}$-universal ensures that it is a submersion near $(0,0)$ (for a proof of this fact, see [33, Proposition 14.3]). Thus, by smoothness with respect to ε, it follows that, for small fixed $\varepsilon\neq 0$, $(x,\mu)\mapsto\Delta_\ell(x,\mu,\varepsilon)$ is also a submersion near the origin. Hence, $\mathcal{D}_{\ell,0}$ and $\mathcal{D}_\varepsilon$ are smooth manifolds of codimension k by the Regular Value Theorem.

The fact that $\mathcal{D}_\varepsilon$ is $\mathcal{O}(\varepsilon)$-close to $\mathcal{D}_{\ell,0}$ follows from Theorem 1. In fact, since $\mathcal{D}_\varepsilon$ can be obtained, for $\varepsilon\neq 0$, by intersecting M_Π with the hyperplane attained by fixing ε, it follows that that $\mathcal{D}_\varepsilon$ is given by the image under Φ of $Z_{g_\ell}\times\{\Phi_3^{-1}(\varepsilon)\}$. Thus, if $\varepsilon' := \Phi_3^{-1}(\varepsilon)$,

$$\mathcal{D}_\varepsilon = \{(\Phi_1(x,\mu,\varepsilon'),\Phi_2(\mu,\varepsilon')) : (x,\mu,\varepsilon')\in\left(Z_{g_\ell}\times\mathbb{R}\cap U\right)\} \tag{94}$$

Considering that, by definition, $\varepsilon' = \mathcal{O}(\varepsilon)$ and that Φ is smooth, if follows that $\mathcal{D}_\varepsilon$ is $\mathcal{O}(\varepsilon)$-close to

$$\{(\Phi_1(x,\mu,0),\Phi_2(\mu,0)) : (x,\mu)\in Z_{g_\ell}\}, \tag{95}$$

which coincides with $Z_{g_\ell} = \mathcal{D}_{\ell,0}$ by the invariance statement of Theorem 1. This concludes the proof.

5.4. **Proof of stabilisation of non-stable families: the transcritical case.** For a 1-dimensional vector field, the transcritical bifurcation is generally described as occurring in a 1-parameter family, as two equilibria collide and pass through each other, exchanging their stability properties. A normal form for the transcritical bifurcation is $\dot{x} = \mu x + x^2$.

Families displaying such behaviour are not "stable", in that a small perturbation thereof generally changes the phase portraits and breaks the bifurcation. However, they are still studied because they appear "generically" in 1-parameter families displaying a fairly common property: existence of an equilibrium for every value of the parameter (see [16, Section 3.4]).

In this section, we focus our attention in 1-dimensional vector fields undergoing a transcritical bifurcation appearing in the guiding system to exemplify the phenomenon of stabilisation that the inclusion of the perturbation parameter has. To do so, we provide a definition of the transcritical bifurcation based on the concept of $\mathcal{K}$-equivalence.

Definition 21. *A 1-parameter family of* 1*-dimensional vector fields $F(x, \mu)$ is said to undergo a transcritical bifurcation at the origin for $\mu = 0$ if*

1. *The germ of $f : x \mapsto F(x, 0)$ at the origin is $\mathcal{K}$-equivalent to the germ of $s_{1,0}(x) = x^2$.*
2. *Let $([M], [\phi]) \in GL_n(\mathcal{E}_n) \times L_n$ be such that $[f] = [M] \cdot [s_{1,0}] \circ [\phi]$. The pushforward $([M], [\phi]) * [\tilde{\mathcal{U}}]$ of the unfolding $[\mathcal{U}] \in \mathcal{Z}^1_{1,1}$, given by $\tilde{\mathcal{U}}(x, \mu) = (\mathcal{U}(x, \mu), \mu)$ and $\mathcal{U}(x, \mu) = \mu x + x^2$, is $\mathcal{K}$-equivalent to $[\tilde{F}]$ via the identity, where $\tilde{F}(x, \mu) = (F(x, \mu), \mu)$.*

The definition essentially states that a transcritical family is characterized by a singularity whose unfolding is, up to $\mathcal{K}$-equivalence, given by the normal form $x \mapsto \mu x - x^2$.

5.4.1. *The canonical form of the displacement function.*

Proposition 10. *Let $n = k = 1$ and suppose that the guiding system $\dot{x} = g_\ell(x, \mu)$ undergoes a transcritical bifurcation at the origin for $\mu = 0$. Then, there are $\varepsilon_1 \in (0, \varepsilon_0)$, an open interval I containing $0 \in \mathbb{R}$, an open neighbourhood $U_\Sigma \subset \Sigma$ of 0, and smooth functions $\zeta, Q : I \times U_\Sigma \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$, $a, S : U_\Sigma \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$, and $b : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ such that*

(T.I) If Δ_ℓ is the displacement function of order ℓ of (1)*, then*

$$\Delta_\ell(x, \mu, \varepsilon) = Q(x, \mu, \varepsilon) \left(\zeta^2(x, \mu, \varepsilon) + S(\mu, \varepsilon) a^2(\mu, \varepsilon) + S(\mu, \varepsilon) b(\varepsilon) \right)$$

for $(x, \mu, \varepsilon) \in I \times U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$.

(T.II) For each $(\mu, \varepsilon) \in U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$, the map $\zeta_{(\mu,\varepsilon)} : x \mapsto \zeta(x, \mu, \varepsilon)$ is a diffeomorphism on the interval I.

(T.III) For each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $a_\varepsilon : \mu \mapsto a(\mu, \varepsilon)$ is a diffeomorphism on U_Σ.

(T.IV) $b(0) = 0$, $a(0, 0) = 0$, $\zeta(0, 0, 0) = 0$, and $\operatorname{sign}(Q(0, 0, 0)) = \operatorname{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0, 0)\right)$.

(T.V) $S(\mu, \varepsilon) < 0$ for any $(\mu, \varepsilon) \in U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$.

Proof. We begin by observing that $\Delta_\ell(x, \mu, 0) = T g_\ell(x, \mu)$, by definition of the displacement function of order ℓ. Let $[s_{1,0}]$ be as in Definition 21 and $[\tilde{F}]$ be the 2-parameter unfolding of $f : x \mapsto T g_\ell(x, 0)$ given by $\tilde{F}(x, \mu, \varepsilon) = (\Delta_\ell(x, \mu, \varepsilon), \mu, \varepsilon)$. By hypothesis, there are $P(x, \mu) \in \mathbb{R}$ and $\psi(x, \mu) \in \mathbb{R}$ such that

$$\Delta_\ell(x, \mu, 0) = P(x, \mu) \left(\mu \psi(x, \mu) + \psi^2(x, \mu) \right) \tag{96}$$

and, defining $M(x) := P(x, 0)$, and $\phi(x) := \psi(x, 0)$, it holds that $[f] = [M] \cdot [s_{1,0}] \circ [\phi]$.

Let $\tilde{H}(x,\eta) = (y^2+\eta,\eta) \in \mathbb{R}\times\mathbb{R}$. Since the 1-parameter unfolding $[\tilde{H}]$ of $[s_{1,0}]$ is $\mathcal{K}$-versal, it follows that $[\tilde{F}]$ must be $\mathcal{K}$-induced by $([M],[\phi]) * [\tilde{H}]$. Hence, there is a neighbourhood $\tilde{V}_1 := I \times (-\tilde{\mu}_1,\tilde{\mu}_1)\times(-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)$ of the origin in $\mathbb{R}^{1+1+1}$ and smooth functions $h(\mu,\varepsilon)\in\mathbb{R}$, $Q(x,\mu,\varepsilon)\in\mathbb{R}$, and $\zeta(x,\mu,\varepsilon)\in\mathbb{R}$ such that $h(0,0)=0$, $Q(x,0,0)=M(x)\neq 0$, $\zeta(x,0,0)=\phi(x)$, and

$$\Delta_\ell(x,\mu,\varepsilon) = Q(x,\mu,\varepsilon)\cdot\Big(\zeta^2(x,\mu,\varepsilon)+h(\mu,\varepsilon)\Big). \tag{97}$$

Because $\zeta(x,0,0)=\phi(x)$ is a local diffeomorphism, assuming that $\tilde{\mu}_1$ and $\tilde{\varepsilon}_1$ are sufficiently small, we can ensure that $\zeta_{(\mu,\varepsilon)} : x\mapsto\zeta(x,\mu,\varepsilon)$ is a diffeomorphism on I for any $(\mu,\varepsilon)\in(-\tilde{\mu}_1,\tilde{\mu}_1)\times(-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)$.

Since $[f]=[M]\cdot[s_{1,0}]\circ[\phi]$, it follows by twice differentiating at the origin that

$$T\frac{\partial^2 g_\ell}{\partial x^2}(0,0) = 2M(0)\left(\phi'(0)\right)^2. \tag{98}$$

Considering that $M(x)=Q(x,0,0)$, we obtain

$$\operatorname{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0,0)\right) = \operatorname{sign}(M(0)) = \operatorname{sign}\left(Q(0,0,0)\right)\neq 0. \tag{99}$$

A combination of (96) and (97) yields

$$P(x,\mu)\Big(\mu\psi(x,\mu)+\psi^2(x,\mu)\Big) = Q(x,\mu,0)\cdot\Big(\zeta^2(x,\mu,0)+h(\mu,0)\Big) \tag{100}$$

Differentiating both sides of (100) with respect to μ at the origin and considering that $\psi(0,0)=\zeta(0,0,0)=\phi(0)=0$, it follows that

$$\frac{\partial h}{\partial\mu}(0,0)=0. \tag{101}$$

Now, differentiating both sides of (100) twice with respect to x at the origin and considering that $M(0)$ is invertible, we obtain

$$\left(\frac{\partial\psi}{\partial x}(0,0)\right)^2 = \left(\frac{\partial\zeta}{\partial x}(0,0,0)\right)^2. \tag{102}$$

On the other hand, a mixed differentiation with respect to x and μ yields:

$$\frac{\partial\psi}{\partial x}(0,0)+2\left(\frac{\partial\psi}{\partial\mu}(0,0)\right)\left(\frac{\partial\psi}{\partial x}(0,0)\right) = 2\left(\frac{\partial\zeta}{\partial x}(0,0,0)\right)\left(\frac{\partial\zeta}{\partial\mu}(0,0,0)\right). \tag{103}$$

Finally, differentiating both sides of (100) twice with respect to μ and considering (101), we obtain

$$2\frac{\partial\psi}{\partial\mu}(0,0)+2\left(\frac{\partial\psi}{\partial\mu}(0,0)\right)^2 = 2\left(\frac{\partial\zeta}{\partial\mu}(0,0,0)\right)^2+\frac{\partial^2 h}{\partial\mu^2}(0,0). \tag{104}$$

Squaring (103) and considering (102), it follows that

$$1+4\frac{\partial\psi}{\partial\mu}(0,0)+4\left(\frac{\partial\psi}{\partial\mu}(0,0)\right)^2 = 4\left(\frac{\partial\zeta}{\partial\mu}(0,0,0)\right)^2. \tag{105}$$

Hence, combining with (104), we obtain

$$\frac{\partial^2 h}{\partial\mu^2}(0,0) = -\frac{1}{2}. \tag{106}$$

Considering (101) and (106), it follows from Taylor's theorem that $h(\mu,0) = \mu^2 r(\mu)$, where r is smooth and $r(0) = -\frac{1}{4} < 0$. Hence, it is clear that $[h_0] = [r]\cdot[s_{1,0}]$, and $[h_0]$ is $\mathcal{K}$-equivalent to $[s_{1,0}]$. Thus, as before, it follows that the 1-parameter unfolding $[h]$ of $[h_0]$ must be $\mathcal{K}$-induced by $([r],[\mathrm{Id}]) * [\tilde{H}]$, that is, there are smooth real functions $S(\mu,\varepsilon)$, $a(\mu,\varepsilon)$, and $b(\varepsilon)$, defined on $(-\tilde{\mu}_2,\tilde{\mu}_2)\times(-\tilde{\varepsilon}_2,\tilde{\varepsilon}_2) \subset (-\tilde{\mu}_1,\tilde{\mu}_1)\times(-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)$, such that $S(\mu,0) = r(\mu)$, $a(\mu,0) = \mu$, $b(0) = 0$, and

$$h(\mu,\varepsilon) = S(\mu,\varepsilon)\cdot\Big(a^2(\mu,\varepsilon) + b(\varepsilon)\Big). \tag{107}$$

holds locally near the origin. Since $S(0,0) = r(0) < 0$, we can assume that $\tilde{\mu}_2$ and $\tilde{\varepsilon}_2$ are sufficiently small as to ensure that $S(\mu,\varepsilon) < 0$ for any $(\mu,\varepsilon) \in (-\tilde{\mu}_2,\tilde{\mu}_2)\times(-\tilde{\varepsilon}_2,\tilde{\varepsilon}_2)$. Moreover, they can be assumed sufficiently small to guarantee that a_ε is a diffeomorphism as well. □

5.4.2. *Proof of Theorem 3.* By definition of Δ_ℓ, it is easy to see that

$$\frac{\partial\Delta_\ell}{\partial\varepsilon}(0,0,0) = g_{\ell+1}(0,0), \tag{108}$$

which is non-zero by hypothesis. Let $V := I \times U_\Sigma \times (-\varepsilon_1,\varepsilon_1)$ as given in Proposition 10. Then, item (T.I) ensures that

$$\frac{\partial\Delta_\ell}{\partial\varepsilon}(0,0,0) = Q(0,0,0)S(0,0)b'(0). \tag{109}$$

Thus, considering items (T.IV) and (T.V), it follows that

$$b'(0) = \sigma\frac{g_{\ell+1}(0,0)}{|Q(0,0,0)S(0,0)|}, \tag{110}$$

where

$$\sigma = \mathrm{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0,0)\right) \in \{-1,1\}. \tag{111}$$

Now, item (T.I) also ensures that $\Delta_\ell(x,\mu,\varepsilon) = 0$ is equivalent to

$$b(\varepsilon) = -\frac{1}{S(\mu,\varepsilon)}\zeta^2(x,\mu,\varepsilon) - a^2(\mu,\varepsilon) \tag{112}$$

in V. Define $\Psi(x,\mu,\varepsilon) = (\Psi_1(x,\mu,\varepsilon),\Psi_2(\mu,\varepsilon),\Psi_3(\varepsilon))$ by

$$\Psi_1(x,\mu,\varepsilon) = \frac{\zeta(x,\mu,\varepsilon)}{\sqrt{-S(\mu,\varepsilon)}}, \quad \Psi_2(\mu,\varepsilon) = a(\mu,\varepsilon), \quad \Psi_3(\varepsilon) = b(\varepsilon). \tag{113}$$

Hence, Ψ is a strongly-fibred diffeomorphism onto its image U and (112) is itself equivalent to

$$\Psi_3(\varepsilon) = (\Psi_1(x,\mu,\varepsilon))^2 - (\Psi_2(\mu,\varepsilon))^2. \tag{114}$$

Thus, $\Delta_\ell(x,\mu,\varepsilon) = 0 \iff \Psi(x,\mu,\varepsilon) \in \{(y,\theta,\eta) \in \mathbb{R}^3 : \eta = y^2 - \theta^2\}$. Defining $\Phi = \Psi^{-1}$, it follows that $\Delta_\ell(x,\mu,\varepsilon) = 0 \iff (x,\mu,\varepsilon) \in \Phi\left(\{(y,\theta,\eta) \in \mathbb{R}^3 : \eta = y^2 - \theta^2\}\right)$.

Furthermore, since $\Phi_3(\varepsilon) = b^{-1}(\varepsilon)$, it follows from (110) that

$$\mathrm{sign}\left(\Phi_3'(0)\right) = \sigma\cdot\mathrm{sign}\left(g_{\ell+1}(0,0)\right). \tag{115}$$

Finally, since $\Delta_\ell(x,\mu,0) = Tg_\ell(x,\mu)$, it is easy to see that, if we fix $\varepsilon = 0$, we have $g_\ell(x,\mu) = 0 \iff (\Psi_1(x,\mu,0))^2 = (\Psi_2(\mu,0))^2$, proving that

$$(Z_{g_\ell} \times \{0\}) \cap V = \Phi\left(\left\{(y,\theta,0) \in \mathbb{R}^3 : y^2 - \theta^2 = 0\right\} \cap U\right). \tag{116}$$

5.4.3. *Description of the perturbed bifurcation.* We now make use of the results above to describe the behaviour of Π for values of the parameter near the point of bifurcation. Essentially, we show that, in one direction of variation of ε, the transcritical is broken into two nearby folds, whereas in the other no bifurcation occurs.

We assume, without loss of generality, that

$$\operatorname{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0,0)\right) \operatorname{sign}\left(g_{\ell+1}(0,0)\right) = 1, \tag{117}$$

which is equivalent to assuming the orientation of the saddle obtained for the catastrophe surface in Theorem 3. If this product is negative, the behaviour is analogous, but mirrored with respect to the sign of the perturbation parameter ε.

Proposition 11. *Let $n = 1$ and suppose the guiding system $\dot{x} = g_\ell(x,\mu)$ undergoes a transcritical bifurcation at the origin for $\mu = 0$. Also, let I, U_Σ and ε_1 be as provided in Proposition 10, and define $\sigma = \operatorname{sign}\left(\frac{\partial^2 g_\ell}{\partial x^2}(0,0)\right)$ and $\sigma' = \operatorname{sign}\left(g_{\ell+1}(0,0)\right)$. If $\sigma\sigma' = 1$, there are $(x_2,\mu_2,\varepsilon_2) \in (I \cap \mathbb{R}_+^*) \times (U_\Sigma \cap \mathbb{R}_+^*) \times (0,\varepsilon_1)$ and continuous functions $\mu_c, \mu_e : (-\varepsilon_2,\varepsilon_2) \to (-x_2,x_2)$ such that the following hold:*

- *(a) For each $\varepsilon \in (-\varepsilon_2, 0)$, the family $(x,\mu) \mapsto \Pi(x,\mu,\varepsilon)$ undergoes two fold-like bifurcations in the set $(-x_2,x_2)$ as μ traverses $(-\mu_2,\mu_2)$, one at $\mu = \mu_e(\varepsilon) \in (0,\mu_2)$ and another at $\mu = \mu_c(\varepsilon) \in (-\mu_2,0)$. In other words, if we take μ to grow through $(-\mu_2,\mu_2)$, we observe the collision of two hyperbolic fixed points as $\mu = \mu_c(\varepsilon)$ and the subsequent emergence of two hyperbolic fixed points at $\mu = \mu_e(\varepsilon)$. When $\mu = \mu_c(\varepsilon)$ or $\mu = \mu_e(\varepsilon)$, there is one fixed point that is nonhyperbolic. Apart from those mentioned, there are no other fixed points in the interval $(-x_2,x_2)$. In particular, there are no fixed points in this interval for $\mu \in (-\mu_c(\varepsilon),\mu_e(\varepsilon))$.*
- *(b) For each $\varepsilon \in (0,\varepsilon_2)$, the family $(x,\mu) \mapsto \Pi(x,\mu,\varepsilon)$ does not undergo any bifurcation in $(-x_2,x_2)$ as μ traverses $(-\mu_2,\mu_2)$. If we take μ to grow past this interval, we observe exactly two hyperbolic fixed points in $(-x_2,x_2)$, first approaching without colliding, and then straying apart.*

Proof. Take $\tilde{\varepsilon}_1 := \varepsilon_1$, $\tilde{x}_1, \tilde{\mu}_1 > 0$ such that $(-\tilde{x}_1,\tilde{x}_1) \subset I$, and $[-\tilde{\mu}_1,\tilde{\mu}_1] \subset U_\Sigma$ and define $W_1 = (-\tilde{x}_1,\tilde{x}_1) \times (-\tilde{\mu}_1,\tilde{\mu}_1) \times (-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)$. In that case, item (T.I) ensures that

$$\Delta_\ell(x,\mu,\varepsilon) = Q(x,\mu,\varepsilon)\left(\zeta^2(x,\mu,\varepsilon) + S(\mu,\varepsilon)a^2(\mu,\varepsilon) + S(\mu,\varepsilon)b(\varepsilon)\right), \tag{118}$$

for $(x,\mu,\varepsilon) \in W_1$.

Let $\Lambda : (-\tilde{\mu}_1,\tilde{\mu}_1) \times (-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1) \to \mathbb{R}^2$ be given by $\Lambda(\mu,\varepsilon) = (a(\mu,\varepsilon),\varepsilon)$. Since a_ε is a diffeomorphism on U_Σ for $\varepsilon \in (-\tilde{\varepsilon}_1,\tilde{\varepsilon}_1)$, it follows that Λ is a diffeomorphism onto its image. Considering that $a(0,0) = 0$, $E_\Lambda := \operatorname{Im}\Lambda$ is an open set containing $(0,0) \in \mathbb{R}^2$. Thus, there is a basic open neighbourhood of the origin $(-\tilde{a},\tilde{a}) \times (-\tilde{\varepsilon}_2,\tilde{\varepsilon}_2) \subset E_\Lambda$. This means that $(-\tilde{a},\tilde{a}) \subset \operatorname{Im} a_\varepsilon$ for any $\varepsilon \in (-\tilde{\varepsilon}_2,\tilde{\varepsilon}_2)$. Since $b(0) = 0$ and b is smooth, we can take $\tilde{\varepsilon}_3 \in (0,\tilde{\varepsilon}_2)$ such that $\sqrt{|b(\varepsilon)|} < \tilde{a}$ for any $\varepsilon \in (-\tilde{\varepsilon}_3,\tilde{\varepsilon}_3)$. This ensures that $a_\varepsilon^{-1}\left(\pm\sqrt{|b(\varepsilon)|}\right)$

is well defined for $\varepsilon \in (-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3)$. Hence, we can define

$$\mu_c(\varepsilon) := a_\varepsilon^{-1}\left(-\sqrt{|b(\varepsilon)|}\right) \quad \text{and} \quad \mu_e(\varepsilon) := a_\varepsilon^{-1}\left(\sqrt{|b(\varepsilon)|}\right), \tag{119}$$

both clearly continuous on $(-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3)$ and whose image lies in $(-\tilde{\mu}_1, \tilde{\mu}_1)$.

Proceeding just as in the proof of Theorem 3, we obtain

$$b'(0) = \sigma \frac{g_{\ell+1}(0,0)}{|Q(0,0,0)S(0,0)|}, \tag{120}$$

which does not vanish by hypothesis. Hence, there is $\tilde{\varepsilon}_4 \in (0, \tilde{\varepsilon}_3)$ sufficiently small such that $\operatorname{sign}(b(\varepsilon)) = \sigma\sigma'$ for $\varepsilon \in (0, \tilde{\varepsilon}_4)$ and $\operatorname{sign}(b(\varepsilon)) = -\sigma\sigma'$ for $\varepsilon \in (-\tilde{\varepsilon}_4, 0)$. Henceforth in the proof, we assume, without loss of generality, that $\sigma\sigma' = 1$. The other case can be treated analogously and will be omitted for the sake of brevity.

Now, from (118), it follows that, for $(x, \mu, \varepsilon) \in W_1$, it holds that $\Delta_\ell(x, \mu, \varepsilon) = 0$ if, and only if,

$$\left(\frac{\zeta(x,\mu,\varepsilon)}{\sqrt{-S(\mu,\varepsilon)}}\right)^2 = a^2(\mu,\varepsilon) + b(\varepsilon). \tag{121}$$

We will, therefore, study how many roots of the polynomial $z^2 = a^2(\mu, \varepsilon) + b(\varepsilon)$ exist near zero for each $(\mu, \varepsilon) \in (-\tilde{\mu}_1, \tilde{\mu}_1) \times (-\tilde{\varepsilon}_4, \tilde{\varepsilon}_4)$, since they can then be converted via inverse function to values of x satisfying $\Delta_\ell(x, \mu, \varepsilon) = 0$.

We first study item (a), that is, the case $\varepsilon \in (-\varepsilon_4, 0)$, for which the polynomial equation can be rewritten as $z^2 = a^2(\mu, \varepsilon) + |b(\varepsilon)|$. Considering that $|b(\varepsilon)| > 0$ for any $\varepsilon \in (-\tilde{\varepsilon}_4, 0)$, it is easy to see that this equation has exactly two simple roots for $(\mu, \varepsilon) \in (-\tilde{\mu}_1, \tilde{\mu}_1) \times (-\tilde{\varepsilon}_4, 0)$.

Now, we consider item (b), that is, the case $\varepsilon \in (0, \tilde{\varepsilon}_4)$, for which the polynomial equation can be rewritten as

$$z^2 = a^2(\mu,\varepsilon) - |b(\varepsilon)|. \tag{122}$$

It is thus clear that this equation will have two simple real roots if $a^2(\mu, \varepsilon) > |b(\varepsilon)|$, one double real root if $a^2(\mu, \varepsilon) = |b(\varepsilon)|$ and no real roots if $a^2(\mu, \varepsilon) < |b(\varepsilon)|$. In other words, the number of roots depends solely on the sign of the function

$$c_\varepsilon(\mu) = a^2(\mu,\varepsilon) - |b(\varepsilon)|. \tag{123}$$

There are, for each $\varepsilon \in (0, \tilde{\varepsilon}_4)$, exactly two values of $\mu \in (-\tilde{\mu}_1, \tilde{\mu}_1)$ for which $c_\varepsilon(\mu) = a^2(\mu, \varepsilon) - |b(\varepsilon)| = 0$, namely $\mu_c(\varepsilon) = a_\varepsilon^{-1}(-\sqrt{|b(\varepsilon)|})$ and $\mu_e(\varepsilon) = a_\varepsilon^{-1}(\sqrt{|b(\varepsilon)|})$. We proceed by studying the sign of the $c_\varepsilon(\mu)$ for $\mu \in (-\tilde{\mu}_1, \tilde{\mu}_1)$.

To do so, assume first that $a_0'(0) > 0$. Since a_ε is a diffeomorphism on U_Σ for any $\varepsilon \in (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$, smoothness of a ensures that $a_\varepsilon'(0) > 0$ for any $\varepsilon \in (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$. For the same reason, we obtain that $a_\varepsilon'(\mu) > 0$ for any $(\mu, \varepsilon) \in U_\Sigma \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$. Hence, since $[-\tilde{\mu}_1, \tilde{\mu}_1] \subset U_\Sigma$ and $[-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3] \subset (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$, it follows that

$$m := \inf\{a_\varepsilon'(\mu) : (\mu,\varepsilon) \in [-\tilde{\mu}_1, \tilde{\mu}_1] \times (-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3)\} > 0. \tag{124}$$

Moreover, considering that $(-\tilde{\mu}_1, \tilde{\mu}_1) \subset [-\tilde{\mu}_1, \tilde{\mu}_1]$ and that $(-\tilde{\varepsilon}_4, \tilde{\varepsilon}_4) \subset [-\tilde{\varepsilon}_3, \tilde{\varepsilon}_3]$, we get

$$\inf\{a_\varepsilon'(\mu) : (\mu,\varepsilon) \in (-\tilde{\mu}_1, \tilde{\mu}_1) \times (-\tilde{\varepsilon}_4, \tilde{\varepsilon}_4)\} \geq m > 0. \tag{125}$$

This means that a_ε, and consequently also its inverse, is an strictly increasing function on $(-\tilde{\mu}_1, \tilde{\mu}_1)$, which will allow us to fully understand the sign of c_ε.

Firstly, since $-\sqrt{|b(\varepsilon)|} < 0 < \sqrt{|b(\varepsilon)|}$ and a_ε^{-1} is increasing for any $\varepsilon \in (0, \tilde{\varepsilon}_4)$, it follows that

$$\mu_c(\varepsilon) = a_\varepsilon^{-1}\left(-\sqrt{|b(\varepsilon)|}\right) < a_\varepsilon^{-1}(0) < a_\varepsilon^{-1}\left(\sqrt{|b(\varepsilon)|}\right) = \mu_e(\varepsilon). \tag{126}$$

Thus, since a_ε is also increasing for any $\varepsilon \in (0, \tilde{\varepsilon}_4)$, we obtain

$$a_\varepsilon(\mu_c(\varepsilon)) < 0 < a_\varepsilon(\mu_e(\varepsilon)). \tag{127}$$

Therefore, considering that $c'_\varepsilon(\mu) = 2a'_\varepsilon(\mu)a_\varepsilon(\mu)$ and that $a'_\varepsilon > 0$, we conclude that:

$$c'_\varepsilon(\mu_c(\varepsilon)) < 0 < c'_\varepsilon(\mu_e(\varepsilon)), \tag{128}$$

for any $\varepsilon \in (0, \tilde{\varepsilon}_4)$.

We have thus proved that, if $\mu \in (0, \tilde{\varepsilon}_4)$, then $c_\varepsilon(\mu) > 0$ for $\mu \in (-\tilde{\mu}_1, \mu_c(\varepsilon)) \cup (\mu_e(\varepsilon), \tilde{\mu}_1)$ and $c_\varepsilon(\mu) < 0$ for $\mu \in (\mu_c(\varepsilon), \mu_e(\varepsilon))$. As mentioned before, this suffices to prove item (b). □

5.5. **Proof of stabilisation of non-stable families: the pitchfork case.** The pitchfork bifurcation for flows can be roughly described as a 1-parameter family of 1-dimensional vector fields exhibiting the emergence of three equilibria from one persistent one. A model example is the family $\dot{x} = \mu x + x^3$. If this family is perturbed, this behaviour is generally lost, unless some symmetry is assumed for the perturbation term. In fact, the pitchfork bifurcation appears generically in a class of families presenting symmetry (the so-called $\mathbb{Z}_2$-equivariant systems - see [25, Section 7.4.2], for instance).

In this section, we analyse the pitchfork bifurcation in a similar way to what we have done for the transcritical.

Definition 22. *A 1-parameter family of* 1*-dimensional vector fields $F(x, \mu)$ is said to undergo a pitchfork bifurcation at the origin for $\mu = 0$ if*

(1) The germ of $f : x \mapsto F(x, 0)$ at the origin is $\mathcal{K}$-equivalent to the germ of $s_{1^2,0}(x) = x^3$.
(2) Let $([M], [\phi]) \in GL_n(\mathcal{E}_n) \times L_n$ be such that $[f] = [M] \cdot [s_{1^2,0}] \circ [\phi]$. The pushforward $([M], [\phi]) * [\tilde{\mathcal{U}}]$ of the unfolding $[\mathcal{U}] \in \mathcal{Z}^1_{1,1}$, given by $\tilde{\mathcal{U}}(x, \mu) = (\mathcal{U}(x, \mu), \mu)$ and $\mathcal{U}(x, \mu) = \mu x + x^3$, is $\mathcal{K}$-equivalent to $[\tilde{F}]$ via the identity, where $\tilde{F}(x, \mu) = (F(x, \mu), \mu)$.

5.5.1. *The canonical form of the displacement function.*

Proposition 12. *Let $n = k = 1$ and suppose that the guiding system $\dot{x} = g_\ell(x, \mu)$ undergoes a pitchfork bifurcation at the origin for $\mu = 0$. Then, there are $\varepsilon_1 \in (0, \varepsilon_0)$, an open interval I containing $0 \in \mathbb{R}$, an open neighbourhood $U_\Sigma \subset \Sigma$ of 0, and smooth functions $\zeta, Q : I \times U_\Sigma \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$, $a : U_\Sigma \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$, and $b : (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}$ such that*

(P.I) If Δ_ℓ is the displacement function of order ℓ of (1)*, then*

$$\Delta_\ell(x, \mu, \varepsilon) = Q(x, \mu, \varepsilon)\left(\zeta^3(x, \mu, \varepsilon) + a(\mu, \varepsilon)\zeta(x, \mu, \varepsilon) + b(\mu, \varepsilon)\right)$$

for $(x, \mu, \varepsilon) \in I \times U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$.

(P.II) For each $(\mu, \varepsilon) \in U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$, the map $\zeta_{(\mu,\varepsilon)} : x \mapsto \zeta(x, \mu, \varepsilon)$ is a diffeomorphism on the interval I.

(P.III) For each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, $a_\varepsilon : \mu \mapsto a(\mu, \varepsilon)$ is a diffeomorphism on U_Σ.

(P.IV) $b(0, 0) = \frac{\partial b}{\partial \mu}(0, 0) = 0$, $a(0, 0) = 0$, $\zeta(0, 0, 0) = 0$, and $Q(0, 0, 0) \neq 0$.

Proof. Observe that $\Delta_\ell(x,\mu,0) = Tg_\ell(x,\mu)$, by definition. Let $[s_{1^2,0}]$ be as in Definition 22 and $[\tilde{F}]$ be the 2-parameter unfolding of $f : x \mapsto Tg_\ell(x,0)$ given by $\tilde{F}(x,\mu,\varepsilon) = (\Delta_\ell(x,\mu,\varepsilon),\mu,\varepsilon)$. Since $[g_\ell]$ undergoes a pitchfork bifurcation, there are $P(x,\mu) \in \mathbb{R}$ and $\psi(x,\mu) \in \mathbb{R}$ such that

$$\Delta_\ell(x,\mu,0) = P(x,\mu)\left(\mu\psi(x,\mu) + \psi^3(x,\mu)\right) \tag{129}$$

and, defining $M(x) := P(x,0)$, and $\phi(x) := \psi(x,0)$, it holds that $[f] = [M] \cdot [s_{1,0}] \circ [\phi]$.

Let $\tilde{H}(x,\theta,\eta) = (y^3 + \theta y + \eta, \theta, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Since the 2-parameter unfolding $[\tilde{H}]$ of $[s_{1^2,0}]$ is $\mathcal{K}$-versal, then $[\tilde{F}]$ must be $\mathcal{K}$-induced by $([M],[\phi]) * [\tilde{H}]$. Therefore, there is a neighbourhood $\tilde{V}_1 := I \times (-\tilde{\mu}_1, \tilde{\mu}_1) \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$ of the origin in $\mathbb{R}^{1+1+1}$ and smooth functions $h_(\mu,\varepsilon) = (a(\mu,\varepsilon), b(\mu,\varepsilon)) \in \mathbb{R}^2$, $Q(x,\mu,\varepsilon) \in \mathbb{R}$, and $\zeta(x,\mu,\varepsilon) \in \mathbb{R}$ such that $h(0,0) = (0,0)$, $Q(x,0,0) = M(x) \neq 0$, $\zeta(x,0,0) = \phi(x)$, and

$$\Delta_\ell(x,\mu,\varepsilon) = Q(x,\mu,\varepsilon) \cdot \left(\zeta^3(x,\mu,\varepsilon) + a(\mu,\varepsilon)\zeta(x,\mu,\varepsilon) + b(\mu,\varepsilon)\right). \tag{130}$$

Considering that $\zeta(x,0,0) = \phi(x)$ is a local diffeomorphism, if we assume that $\tilde{\mu}_1$ and $\tilde{\varepsilon}_1$ are sufficiently small, we can ensure that $\zeta_{(\mu,\varepsilon)} : x \mapsto \zeta(x,\mu,\varepsilon)$ is a diffeomorphism on I for any $(\mu,\varepsilon) \in (-\tilde{\mu}_1, \tilde{\mu}_1) \times (-\tilde{\varepsilon}_1, \tilde{\varepsilon}_1)$.

Combining (129) and (130), we have

$$P(x,\mu)\left(\mu\psi(x,\mu) + \psi^3(x,\mu)\right) = Q(x,\mu,0) \cdot \left(\zeta^3(x,\mu,0) + a(\mu,0)\zeta(x,\mu,0) + b(\mu,0)\right). \tag{131}$$

Differentiating both sides of (131) with respect to μ at the origin and considering that $\psi(0,0) = \zeta(0,0,0) = \phi(0) = 0$, it follows that

$$\frac{\partial b}{\partial \mu}(0,0) = 0. \tag{132}$$

Now, differentiating both sides of (131), once with respect to x and once with respect to μ, at the origin and considering that $M(0)$ is invertible, we obtain

$$\frac{\partial a}{\partial \mu}(0,0) = \frac{\partial \psi}{\partial x}(0,0) = \phi'(0) \neq 0. \tag{133}$$

We can assume that $\tilde{\mu}_2$ and $\tilde{\varepsilon}_2$ are sufficiently small as to ensure that $\frac{\partial a}{\partial \mu}(\mu,\varepsilon) \neq 0$ for any $(\mu,\varepsilon) \in (-\tilde{\mu}_2, \tilde{\mu}_2) \times (-\tilde{\varepsilon}_2, \tilde{\varepsilon}_2)$, guaranteeing that a_ε is a diffeomorphism. □

5.5.2. *Proof of Theorem 4.* By definition of Δ_ℓ, it is easy to see that

$$\frac{\partial \Delta_\ell}{\partial \varepsilon}(0,0,0) = g_{\ell+1}(0,0), \tag{134}$$

which is does not vanish. Let $V := I \times U_\Sigma \times (-\varepsilon_1, \varepsilon_1)$ as given in Proposition 12. Then, item (P.I) ensures that

$$\frac{\partial \Delta_\ell}{\partial \varepsilon}(0,0,0) = Q(0,0,0)b'(0). \tag{135}$$

Thus, considering item (P.IV), we obtain

$$\frac{\partial b}{\partial \varepsilon}(0,0) = \frac{g_{\ell+1}(0,0)}{Q(0,0,0)}. \tag{136}$$

Item (P.I) yields that $\Delta_\ell(x,\mu,\varepsilon)=0$ is equivalent to

$$b(\mu,\varepsilon)=-\zeta^3(x,\mu,\varepsilon)-a(\mu,\varepsilon)\zeta(x,\mu,\varepsilon) \tag{137}$$

in V. Define $\Psi(x,\mu,\varepsilon)=(\Psi_1(x,\mu,\varepsilon),\Psi_2(\mu,\varepsilon),\Psi_3(\mu,\varepsilon))$ by

$$\Psi_1(x,\mu,\varepsilon)=\zeta(x,\mu,\varepsilon),\quad \Psi_2(\mu,\varepsilon)=a(\mu,\varepsilon),\quad \Psi_3(\mu,\varepsilon)=b(\mu,\varepsilon), \tag{138}$$

a weakly-fibred diffeomorphism onto its image U. We remark that, since $\frac{\partial b}{\partial\mu}(0,0)=0$ by item (P.IV), we also know that Ψ is strongly-fibred to the first order at the origin. Moreover, (137) is equivalent to

$$(\Psi_1(x,\mu,\varepsilon))^3+\Psi_2(\mu,\varepsilon)\Psi_1(x,\mu,\varepsilon)+\Psi_3(\mu,\varepsilon)=0. \tag{139}$$

Therefore, $\Delta_\ell(x,\mu,\varepsilon)=0 \iff \Psi(x,\mu,\varepsilon)\in\{(y,\theta,\eta)\in\mathbb{R}^3: y^3-\theta y+\eta=0\}$. Defining $\Phi=\Psi^{-1}$, the proof is concluded.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix A. Group structure of germs of fibred diffeomorphisms

It is known that germs of local diffemorphisms at a point (see Definition 6) have a well defined operation induced by composition. Hence, we assume without loss of generality that the domains and images of the diffeomorphisms are compatible with composition.

The fact that the composition of two fibred diffeomorphisms is still a fibred diffeomorphism, be it strongly or weakly fibred, amounts to simple calculation, and will be omitted here. The only property of groups that has to be non-trivially verified is the existence of an inverse element in the class of local diffeomorphisms with the same fibration, which amounts to proving that the inverse of a fibred diffeomorphism is itself still fibred.

Let thus Φ be strongly-fibred and let $\Psi:=\Phi^{-1}$, its inverse diffeomorphism. We wish to prove that Ψ is strongly-fibred as well. we begin by proving that Ψ_3 does not depend on x or μ.

To do so, first notice that, since Φ is diffeomorphism, it follows that, for any (x,μ,ε) in its domain, $\det D\Phi(x,\mu,\varepsilon)\neq 0$. Considering that $\Phi(x,\mu,\varepsilon)=(\Phi_1(x,\mu,\varepsilon),\Phi_2(\mu,\varepsilon),\Phi_3(\varepsilon))$, it follows at once by taking into account the block structure of the matrix $D\Phi(x,\mu,\varepsilon)$ that

$$\det\frac{\partial\Phi_1}{\partial x}(x,\mu,\varepsilon)\neq 0,\quad \det\frac{\partial\Phi_2}{\partial\mu}(\mu,\varepsilon)\neq 0,\quad\text{and}\quad \Phi_3'(\varepsilon)\neq 0. \tag{140}$$

Now differentiate the identity $\Psi_3(\Phi_1(x,\mu,\varepsilon),\Phi_2(\mu,\varepsilon),\Phi_3(\varepsilon))=\varepsilon$ with respect to x to obtain

$$\frac{\partial\Psi_3}{\partial x}(\Phi_1(x,\mu,\varepsilon),\Phi_2(\mu,\varepsilon),\Phi_3(\varepsilon))\cdot\frac{\partial\Phi_1}{\partial x}(x,\mu,\varepsilon)=0, \tag{141}$$

which, combined with (140), ensures that $\frac{\partial\Psi_3}{\partial x}$ vanishes identically in its domain.

Differentiating the same identity with respect to μ and considering that $\frac{\partial \Psi_3}{\partial x} = 0$, we obtain :

$$\frac{\partial \Psi_3}{\partial \mu}(\Phi_1(x,\mu,\varepsilon), \Phi_2(\mu,\varepsilon), \Phi_3(\varepsilon)) \cdot \frac{\partial \Phi_2}{\partial \mu}(\mu,\varepsilon) = 0, \tag{142}$$

now ensuring that $\frac{\partial \Psi_3}{\partial x}$ vanishes identically in its domain. Therefore, Ψ_3 depends solely on ε, as we wished to prove.

Finally, differentiating $\Psi_2(\Phi_1(x,\mu,\varepsilon), \Phi_2(\mu,\varepsilon), \Phi_3(\varepsilon)) = \mu$ with respect to x, it follows that

$$\frac{\partial \Psi_2}{\partial x}(\Phi_1(x,\mu,\varepsilon), \Phi_2(\mu,\varepsilon), \Phi_3(\varepsilon)) \cdot \frac{\partial \Phi_1}{\partial x}(x,\mu,\varepsilon) = 0, \tag{143}$$

which proves that Ψ_2 is independent of x, finishing the proof for the strongly-fibred case.

The weakly-fibred case is analogous, and will be omitted for that reason.

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