An update on *that* singularity

Mike R. Jeffrey

University of Bristol, Department of Engineering Mathematics, Woodland Road, Bristol BS8 1UB, UK
mike.jeffrey@bristol.ac.uk

A bridge over troubled flows

Nothing epitomizes the intrigue of piecewise-smooth dynamics like the two-fold singularity. It is incredibly simple to describe — a point where a flow is tangent to a discontinuity threshold from both sides — yet intricate in its dynamics. Its understanding has pushed the boundaries of understanding in piecewise-smooth systems more than any other discontinuity-induced phenomenon.

It took nearly 30 years, from the translation of Filippov's seminal book introducing the two-fold to the english speaking world, to resolving its switching layer behaviour, before we could say that the two-fold singularity was understood. And it *is* now understood, in wonderful detail: its structural and asymptotic stability [**11, 9**], its bifurcations including its local form and the affect of higher orders [**11, 3, 4**], the winding numbers when a flows rotates around it [**6**], the determinacy or determinacy-breaking that occurs when a flow passes through it [**8**], even its extension to multiple switches [**10**].

We now know that the two-fold singularity's structural stability requires nonlinear switching or *hidden* terms, and that it comes in three main flavours, with numerous subclasses between which bifurcations can occur. We know that it is neither an attractor nor a repellor, but an organizing centre, a bi-directional bridge between attracting and repelling sliding on a switching surface, which can lead to the creation of a determinacy-breaking attractor (described as *non-deterministic chaos* in [**3, 8, 1, 2**]).

The developments towards understanding the two-fold singularity can be traced through the papers [**7, 16, 11, 3, 4, 6, 9, 10**]. Attempts to look beyond nonsmooth theory into the effects of regularization, introducing a non-ideal switch that is smooth, noisy, delayed, or hysteretic, have begun in [**18, 13, 14, 12**]. Finally, while attempts to explore its applications in electronics or mechanics have so far been somewhat unsatisfactory, hints of a deeper role in phase randomization can be found in [**15**].

To summarize the story so far, we must begin, of course, with . . .

Definition 1 A *two-fold* is a point $\mathbf{x}_p$ in a system

$$(1)\qquad \dot{\mathbf{x}} = \left\{ \begin{array}{lll} \mathbf{f}^+(\mathbf{x}) & \text{if} & \sigma(\mathbf{x}) > 0 \\ \mathbf{f}^-(\mathbf{x}) & \text{if} & \sigma(\mathbf{x}) < 0 \end{array} \right\} \quad \text{where} \quad \left. \begin{array}{r} \sigma(\mathbf{x}_p) \\ \mathbf{f}^\pm(\mathbf{x}_p)\cdot\nabla\sigma(\mathbf{x}_p) \end{array} \right\} = 0\ ,$$

and with certain non-degeneracy conditions satisfied at $\mathbf{x}_p$, namely $(\mathbf{f}^\pm\cdot\nabla)^2\sigma \neq 0$, $0 \notin \mathbf{f}^\lambda\cdot\nabla\mathbf{x}$, and with transversality of the surfaces $\sigma = 0$, $\mathbf{f}^+\cdot\nabla\sigma = 0$, $\mathbf{f}^-\cdot\nabla\sigma = 0$. We will introduce the combination $\mathbf{f}^\lambda$ below.

The local dynamics depends entirely on two parameters evaluated at $\mathbf{x}_p$,

$$(2)\qquad \nu^+ = \frac{(\mathbf{f}^+\cdot\nabla)(\mathbf{f}^-\cdot\nabla)\sigma}{\sqrt{|(\mathbf{f}^+\cdot\nabla)^2\sigma.(\mathbf{f}^-\cdot\nabla)^2\sigma|}} \qquad \& \qquad \nu^- = \frac{-\,\mathbf{f}^-\cdot\nabla\mathbf{f}^+\cdot\nabla\sigma}{\sqrt{|(\mathbf{f}^+\cdot\nabla)^2\sigma.(\mathbf{f}^-\cdot\nabla)^2\sigma|}}\ ,$$

characterizing the local curvature of the flow. The product $\nu^+\nu^-$ has a simple geometrical interpretation: it quantifies the jump in the vector field between $\mathbf{f}^\pm$ at the singularity.

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Measuring angles from to the '+' or '−' folds respectively, letting $s^{\pm} = \text{sign}(\mathbf{f}^{\pm} \cdot \nabla)^2 \sigma$,

$$\nu^{+}\nu^{-} \;=\; -s^{+}s^{-}\frac{\cot\phi - \cot\theta_{+}^{+}}{\cot\phi - \cot\theta_{+}^{-}} \;=\; -s^{+}s^{-}\frac{\cot\phi + \cot\theta_{-}^{-}}{\cot\phi + \cot\theta_{-}^{+}}\;, \tag{3}$$

where ϕ is the angle between the folds, and θ_j^i is the angle of $\mathbf{f}^i$ from the 'j' fold, measured in the plane spanned by $\mathbf{f}^+$ and $\mathbf{f}^-$, with i and j denoting the labels $+$ or $-$.

The leading order expansion of the two-fold singularity (sometimes called the 'normal form' in a somewhat loose usage of the terminology) is given [**7, 4**] by

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) \;=\; \left\{ \begin{array}{lll} (-x_2, -s^+, \nu^+) & \text{if} & x_1 > 0 \\ (\;x_3\;,\;\nu^-,\;s^-\;) & \text{if} & x_1 < 0 \end{array} \right\} + \left(\mathsf{O}\left(|\mathbf{x}|^2\right), \mathsf{O}\left(|\mathbf{x}|\right), \mathsf{O}\left(|\mathbf{x}|\right)\right)\;, \tag{4}$$

where $s^{\pm} = \text{sign}\left[(\mathbf{f}^{\pm} \cdot \nabla)^2 \sigma(\mathbf{x}_p)\right]$, and in higher dimensions $\dot{x}_{i \geq 4} = \mathsf{O}\left(|\mathbf{x}|\right)$ for $i = 4, 5, \ldots.$

Bifurcation diagrams

Almost everything we understood until the year 2009 could already be found in Filippov's book [**7**], but much of it was presented in the form of unexplained diagrams whose original source is unknown (with their description emerging across [**17, 4, 6**]).

The wealth of information we have on the leading order dynamics (the truncation of (4)) is summarized in the figure below, see [**4, 6**] for detail.

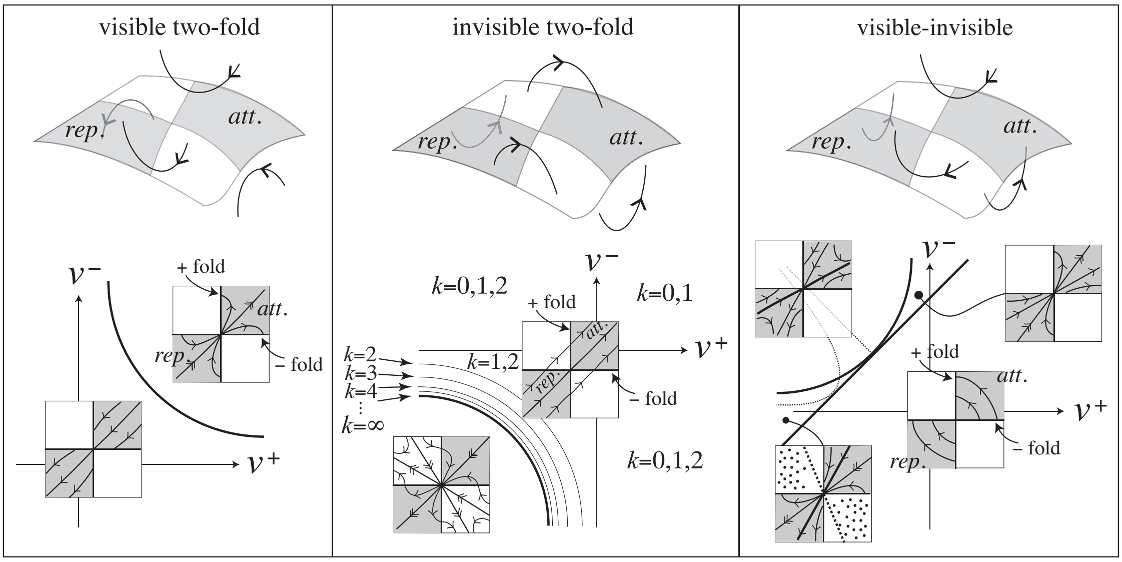

FIGURE 1. Two-folds come in three flavours, formed by the different combinations of visible or invisible folds as determined by the signs of $s^{\pm}$. Top: Regions of attracting sliding (*att.*, shaded), repelling sliding (*rep.*, shaded), and crossing (unshaded) all meet at the singularity. Bottom: Their sliding and crossing topologies in the $\nu^{\pm}$ parameter plane are shown below; for the invisible two-fold, k is the number of windings between visits to the sliding regions, tending to infinity where $\nu^+\nu^- \geq 1$ in $\nu^{\pm} < 0$. See [**4, 6**] for detail.

The folds are:

- both *visible* if $s^+ > 0$ and $s^- < 0$ at $\mathbf{x}_p$,
- both *invisible* if $s^+ < 0$ and $s^- > 0$ at $\mathbf{x}_p$,
- one *visible* and one *invisible* if $s^+s^- > 0$ at $\mathbf{x}_p$,

(we sometimes refer to these as the *flavours* of two-fold).

Crossing maps and winding numbers

The distinguishing feature of the invisible two-fold is that the flow can wind repeatedly around the singularity, making repeated visits to the crossing regions, possibly between entry/exit points to/from the attracting/repelling sliding regions.

Let $\mathbf{y} = (x_2, x_3)$ denote a point on the switching surface $x_1 = 0$, and $\mathbf{y}_i$ denote an iterate of the return map to the switching surface under the flow. A single return to the surface is given by

$$\mathbf{y}_{2m+1} = \underline{\underline{B}}^{\pm}\mathbf{y}_{2m}\ , \qquad \underline{\underline{B}}^{+} = \begin{pmatrix} -1 & 0 \\ -2\nu^{+} & 1 \end{pmatrix} \quad \& \quad \underline{\underline{B}}^{-} = \begin{pmatrix} 1 & -2\nu^{-} \\ 0 & -1 \end{pmatrix}\ , \tag{5}$$

where $\underline{\underline{B}}^{+}$ and $\underline{\underline{B}}^{-}$ are applied in $x_2 < 0$ and $x_3 < 0$ respectively. The second return map, on $x_2 < 0$ or $x_3 < 0$, is therefore

$$\mathbf{y}_{2m+2} = \underline{\underline{A}}^{\pm}\mathbf{y}_{2m}\ , \qquad \underline{\underline{A}}^{\pm} = \underline{\underline{B}}^{\mp}\underline{\underline{B}}^{\pm}\ . \tag{6}$$

Because the maps are associated with folds, they are involutions, so $(\underline{\underline{B}}^{+})^2 = (\underline{\underline{B}}^{-})^2 = \underline{\underline{1}}$ and $\underline{\underline{A}}^{+} = (\underline{\underline{A}}^{-})^{-1}$. The solutions to the difference equation (6) are now obviously

$$\mathbf{y}_{2m} = (\underline{\underline{A}}^{+})^m\mathbf{y}_0 \qquad \text{or} \qquad \mathbf{y}_{2m} = (\underline{\underline{A}}^{-})^m\mathbf{y}_0\ , \tag{7}$$

and a little trigonometry using the substitution $\nu^+\nu^- = \cos^2\Theta$ provides

$$(\underline{\underline{A}}^{\pm})^m = \frac{\sin[2m\Theta]}{\sin 2\Theta}\underline{\underline{A}}^{\pm} - \frac{\sin[2(m-1)\Theta]}{\sin 2\Theta}\underline{\underline{1}}\ . \tag{8}$$

This is also the source of the crossing numbers k in the previous figure. The main dynamical features revealed by the map are shown in the figure below.

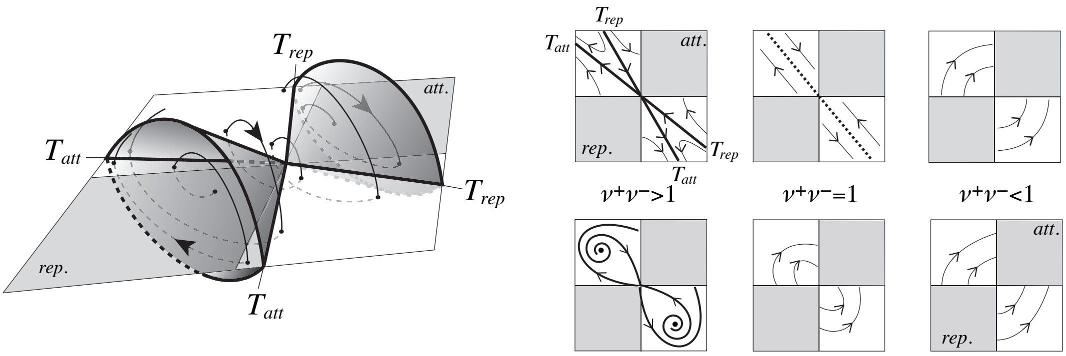

FIGURE 2. The nonsmooth diabolo: invariant manifold (left) around an invisible two-fold. Right top: shown in the switching plane, the manifold bifurcates and disappears at $\nu^+\nu^- = 1$, see **[11]**. Right bottom: the effect of higher order terms, showing a particular case leading to a determinacy-breaking attractor — as the flow exits the repelling sliding region, the crossing flow wraps it back around (via k windings) into the attracting sliding region, whereupon the sliding flow re-injects it back into the repelling region; when all local trajectories pass through the singularity, determinacy is broken; see **[3]**.

Sliding dynamics and hidden instability

To derive sliding dynamics we need to define a *combination* of $\mathbf{f}^{\pm}$ on the switching surface. It turns out that Filippov's combination hides a structural instability, in

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3) = \tfrac{1}{2}(1+\lambda)(-x_2, -s^+, \nu^+) + \tfrac{1}{2}(1-\lambda)(x_3, \nu^-, s^-)\ , \tag{9}$$

essentially because the value $\lambda = \frac{x_3 - x_2}{x_3 + x_2}$ for which sliding occurs is singular at $x_2 = x_3 = 0$.

It is shown in [**9**] that a structurally stable combination is

$$(10)\qquad (\dot{x}_1,\dot{x}_2,\dot{x}_3)=\tfrac{1}{2}\left(1+\lambda\right)\left(-x_2,-s^+,\nu^+\right)+\tfrac{1}{2}\left(1-\lambda\right)\left(x_3,\nu^-,s^-\right)+(1-\lambda^2)(\alpha,0,0)$$

for small $\alpha\neq 0$. A well-defined manifold $\mathcal{M}$ of sliding solutions then exists,

$$(11)\qquad \mathcal{M}=\left\{\ (\lambda,x_2,x_3)\ :\ \tfrac{1}{2}\left(1-\lambda\right)x_3-\tfrac{1}{2}\left(1+\lambda\right)x_2+\alpha(1-\lambda^2)=0\ \right\}\ ,$$

inside the layer $(\lambda,x_2,x_3)\in(-1,+1)\times\mathbb{R}^2$, with $\mathcal{M}$ normally hyperbolic except on

$$(12)\qquad \mathcal{L}=\left\{(\lambda,x_2,x_3)\subset\mathcal{M}:\ \lambda=2\frac{2\alpha+x_3-x_2}{x_3+x_2}=-\frac{x_3+x_2}{4\alpha}\right\}\ ,$$

which corresponds to the two-fold magnified inside the switching layer $\lambda\in(-1,+1)$, $(x_2,x_3)\in\mathbb{R}^2$. The dynamics inside the layer is given by

$$(13)\qquad (\varepsilon\dot{\lambda},\dot{x}_2,\dot{x}_3)=\tfrac{1}{2}\left(1+\lambda\right)\left(-x_2,-s^+,\nu^+\right)+\tfrac{1}{2}\left(1-\lambda\right)\left(x_3,\nu^-,s^-\right)+(1-\lambda^2)(\alpha,0,0)\ ,$$

for $\varepsilon\to 0$, which can be transformed into the well-known singularity of folded slow-manifolds associated with canards in smooth slow-fast systems,

$$(\varepsilon\dot{x},\dot{y},\dot{z})=(y+x^2,pz+qx,r)+\left(\mathsf{O}\left(\varepsilon x,\varepsilon z,xz\right),\mathsf{O}\left(z^2,xz\right),\mathsf{O}\left(z,x\right)\right)$$

provided $\alpha\neq 0$, where p,q,r, are real constants, and provided the conditions $\frac{1}{2}(\nu^+-\nu^-)\leq 1=-s^+=s^-$ or $\frac{1}{2}(\nu^+-\nu^-)\geq -1=-s^+=s^-$ do not hold.

References

[1] M. Buchanan. Differentiating the discontinuous. *Nature Physics - Thesis*, 7:589, 2011.

[2] M. Buchanan. Generating chaos in a new way. *Phys. Rev. Focus*, 28:1, 2011.

[3] A. Colombo and M. R. Jeffrey. Non-deterministic chaos, and the two-fold singularity in piecewise smooth flows. *SIAM J. App. Dyn. Sys.*, 10:423–451, 2011.

[4] A. Colombo and M. R. Jeffrey. The two-fold singularity: leading order dynamics in n-dimensions. *Physica D*, 263:1–10, 2013.

[5] F. Dumortier, R. Roussarie, J. Sotomayor, and H. Zoladek. *Bifurcations of Planar Vector Fields: Nilpotent Singularities and Abelian Integrals*. Springer-Verlag, 1991.

[6] S. Fernández-Garcia, D. Angulo-Garcia, G. Olivar-Tost, M. di Bernardo, and M. R. Jeffrey. Structural stability of the two-fold singularity. *SIAM J. App. Dyn. Sys.*, 11(4):1215–1230, 2012.

[7] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publ. Dortrecht, 1988 (Russian 1985).

[8] M. R. Jeffrey. Non-determinism in the limit of nonsmooth dynamics. *PRL*, 106(25):254103, 2011.

[9] M. R. Jeffrey. Hidden degeneracies in piecewise smooth dynamical systems. *Int. J. Bif. Chaos*, 26(5):1650087(1–18), 2016.

[10] M. R. Jeffrey. Exit from sliding in piecewise-smooth flows: deterministic vs. determinacy-breaking. *Chaos*, 26(3):033108:(1-20), 2016.

[11] M. R. Jeffrey and A. Colombo. The two-fold singularity of discontinuous vector fields. *SIAM Journal on Applied Dynamical Systems*, 8(2):624–640, 2009.

[12] K. U. Kristiansen and S. J. Hogan. Regularizations of two-fold bifurcations in planar piecewise smooth systems using blow up. *submitted*, 2015.

[13] J. Llibre, P. R. da Silva, and M. A. Teixeira. Sliding vector fields via slow-fast systems. *Bull. Belg. Math. Soc. Simon Stevin*, 15(5):851–869, 2008.

[14] D. J. W. Simpson. On resolving singularities of piecewise-smooth discontinuous vector fields via small perturbations. *Discrete Contin. Dyn. Syst. to appear*, 2014.

[15] D. J. W. Simpson and M. R. Jeffrey. Fast phase randomisation via two-folds. *submitted*, 2015.

[16] M. A. Teixeira. Stability conditions for discontinuous vector fields. *J. Differ. Equ.*, 88:15–29, 1990.

[17] M. A. Teixeira. Generic bifurcation of sliding vector fields. *J. Math. Anal. Appl.*, 176:436–457, 1993.

[18] M. A. Teixeira, J. Llibre, and P. R. da Silva. Regularization of discontinuous vector fields on R^3 via singular perturbation. *Journal of Dynamics and Differential Equations*, 19(2):309–331, 2007.

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